

Statistical Mechanics of Polymer Networks of Any Topology

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The statistical mechanics is considered of any polymer network with a prescribed topology, in dimension d , which was introduced previously. The basic direct renormalization theory of the associated continuum model is established. It has a very simple multiplicative structure in terms of the partition functions of the star polymers constituting the vertices of the network. A calculation is made to $O(\varepsilon^2)$, where $d = 4 - \varepsilon$, of the basic critical dimensions σ_L associated with any L -leg vertex ($L \geq 1$). From this infinite series of critical exponents, any topology-dependent critical exponent can be derived. This is applied to the configuration exponent $\gamma_{\mathcal{G}}$ of any network \mathcal{G} to $O(\varepsilon^2)$, including L -leg star polymers. The infinite sets of contact critical exponents θ between multiple points of polymers or between the cores of several star polymers are also deduced. As a particular case, the three exponents $\theta_0, \theta_1, \theta_2$ calculated by des Cloizeaux by field-theoretic methods are recovered. The limiting exact logarithmic laws are derived at the upper critical dimension $d = 4$. The results are generalized to the series of topological exponents of polymer networks near a surface and of tricritical polymers at the Θ -point. Intersection properties of networks of random walks can be studied similarly. The above factorization theory of the partition function of any polymer network over its constituting L -vertices also applies to two dimensions, where it can be related to conformal invariance. The basic critical exponents σ_L and thus any topological polymer exponents are then exactly known. Principal results published elsewhere are recalled.

KEY WORDS: Polymer networks; star polymers; self-avoiding walks; multiplicative renormalization; critical exponents; ε expansion; $O(n)$ model; conformal invariance; two dimensions; Θ -solvent; surface critical behavior.

1. INTRODUCTION

In a previous letter,⁽¹⁾ I have shown that the polymer theory embodies an infinite series of topological (or geometrical) critical exponents, which

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appear in a natural way when studying polymer networks of arbitrary and fixed topology. These exponents were calculated to first order in ε , where $d = 4 - \varepsilon$ is the space dimension, and *exactly in two dimensions*. The aim of the present paper is to describe in more detail the scaling and direct renormalization theory of polymer networks, and in particular to calculate all geometrical exponents to second order, $O(\varepsilon^2)$.

It is well known that long polymer chains in a good solvent constitute a *critical* system. This was originally recognized through their equivalence to a magnetic n -component spin model, with $O(n)$ symmetry, and in the limit $n \rightarrow 0$.⁽²⁾ This made it possible to obtain immediately the size and configuration exponents ν and γ such that

$$R^2 \sim N^{2\nu} \quad (N \gg 1) \quad (1.1)$$

$$\mathcal{Z} \sim \mu^N N^{\gamma-1} \quad (N \gg 1) \quad (1.2)$$

where R^2 is the averaged square end-to-end distance of a single chain of N monomers and \mathcal{Z} is its configuration number. μ is the nonuniversal effective connectivity constant, which is model (i.e., lattice) dependent. In the magnetic $n=0$ model, ν and γ are the universal correlation length and susceptibility exponents. Their Wilson-Fisher ε expansion in space dimension $d = 4 - \varepsilon$ is then immediately obtained from the φ^4 field theory results in the limit $n=0$ (see, e.g., refs. 3 and 4; for orders up to ε^5 see ref. 4)

$$\begin{aligned} \nu &= \frac{1}{2} \left\{ 1 + \frac{\varepsilon}{8} + \frac{15}{4} \left(\frac{\varepsilon}{8} \right)^2 + \left[\frac{135}{8} - 33\zeta(3) \right] \left(\frac{\varepsilon}{8} \right)^3 + \dots \right\} \\ \gamma &= 1 + \frac{\varepsilon}{8} + \frac{13}{4} \left(\frac{\varepsilon}{8} \right)^2 + \left[\frac{97}{8} - 33\zeta(3) \right] \left(\frac{\varepsilon}{8} \right)^3 + \dots \end{aligned} \quad (1.3)$$

Direct renormalization methods (not resorting to field theory) were later devised specifically for polymers, either in a good solvent^(5,6) or in a Θ -solvent.⁽⁷⁾ However, in general, these studies concentrate on the usual critical exponents ν , γ , the crossover exponent ω ,^(3,5) and the various universal scaling functions, which are obtained only at first order in ε .⁽⁸⁾

A noticeable exception is the calculation⁽⁹⁾ at $O(\varepsilon^2)$ by field-theoretic methods of *contact* exponents θ_0 , θ_1 , θ_2 . These exponents describe the short-distance scaling behavior of the probability $P(r)$ that two fixed points of a linear polymer chain approach each other at a distance r . The θ_0 , θ_1 , θ_2 correspond, respectively, to the contact of the two endpoints of a chain [$\theta_0 = (\gamma - 1)/\nu$],⁽⁹⁾ of one end with an interior point of the chain (θ_1), and of two interior points (θ_2). θ_1 and θ_2 are new geometrical exponents, which are *independent* of ν and γ .

But many more questions can be asked in polymer physics. First, one can consider a single star polymer^(1,10) \mathcal{S}_L with L legs of N monomers each, N being large (Fig. 1). Then its number of self-avoiding configurations is a generalization of (1.2),

$$\mathcal{Z}(\mathcal{S}_L) \sim \mu^{LN} N^{\gamma_L - 1} \quad (N \gg 1) \tag{1.4}$$

where μ is the same local connectivity constant as in (1.2) for a single chain. The single-chain case corresponds to $L = 1$, and $\gamma_1 \equiv \gamma$. For $L = 2$ one has a two-leg star, which is still a single linear polymer, hence $\gamma_2 = \gamma$. But for $L \geq 3$, the γ_L constitute a new set of *independent* critical exponents, which are to be determined.

Also, one can wonder about the probability P that the cores of two star polymers with L and L' branches approach each other at a distance r (Fig. 1). At short distance the cores repel each other and the probability vanishes like

$$P(r) \underset{r \rightarrow 0}{\sim} r^{\theta_{L,L'}} \tag{1.5}$$

where $\theta_{L,L'}$ is a new universal *contact exponent* depending only on L and L' , and on the space dimension d .

So a new physics begins which is related to high-order vertices,^(1,2) to star polymers,⁽¹⁰⁾ or to high-contact exponents,⁽¹¹⁾ or also to topological networks.^(1,12) For instance, the contact exponents θ_1, θ_2 described above embody, respectively, the anomalous dimensions of $L = 3$ and $L = 4$ vertices.^(9,11) All the questions above can be answered by unifying all the polymer theory in terms of a *single infinite series* of independent and

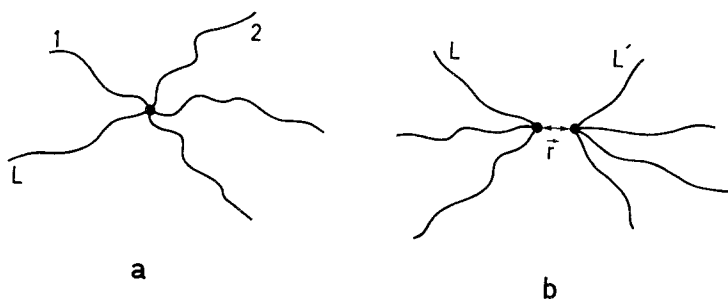


Fig. 1. (a) An L -leg uniform star polymer with equal arms of length S . Its number of configurations scales like $\mathcal{Z}(\mathcal{S}_L) \sim \mu^{LS} S^{\gamma_L - 1}$ for $S \rightarrow \infty$, where γ_L is a universal exponent depending only on L (and d). (b) Two L and L' stars approaching each other at a relative distance r , the probability scaling like $P_{L,L'}(r) \sim r^{\theta_{L,L'}}$ for $r \rightarrow 0$, where $\theta_{L,L'}$ is universal, depending only on L, L' (and d).

universal critical exponents^(1,5) σ_L indexed by an integer $L \geq 1$. They correspond to the new anomalous critical dimensions^(1,12) associated with the physical branching points of L polymers tied together (Fig. 2). In other words, only vertices matter in polymer theory, and one can (almost) forget about the polymer lines connecting them! As we shall see, each core of a star polymer has its own dimension depending only on L , the number of legs of the L -vertex. $L=1$ corresponds to the single *extremity* of a linear polymer and is associated with the usual magnetic exponent η (or γ), $L=2$ corresponds to a *two-leg* vertex, hence to a single monomer inside a polymer chain, and its anomalous dimension is simply associated with the fractal dimension exponent ν . For $L \geq 3$, the σ_L are new critical exponents.

All geometrical critical exponents like γ_L in (1.4) or $\theta_{L,L'}$ in (1.5) can be expressed in terms of the σ_L . For example, the “magnetic” enhancement exponent γ_L for a star reads simply⁽¹⁾

$$\gamma_L - 1 = \sigma_L + L\sigma_1 \tag{1.6}$$

This is very easy to understand: the core contributes an exponent σ_L while the $L=1$ extremities in number L contribute each a σ_1 . In a similar way, the two-star contact exponent $\theta_{L,L'}$ in (1.5) reads^(11,13)

$$\theta_{L,L'} = \frac{1}{\nu} (\sigma_L + \sigma_{L'} - \sigma_{L+L'}) \tag{1.7}$$

which is entirely reminiscent of an operator product expansion (OPE) in field theory. Indeed, σ_L and $\sigma_{L'}$ are the exponents of the two cores, while $\sigma_{L+L'}$ is that of the new core made by the fusion of the two stars upon contact (Fig. 2). From the “vertex structure” of the theory, one can extract new scaling relations between exponents. For instance, from (1.6) and (1.7) one can eliminate the σ_L and get

$$\theta_{L,L'} = \frac{1}{\nu} (\gamma_L + \gamma_{L'} - \gamma_{L+L'} - 1) \tag{1.8}$$

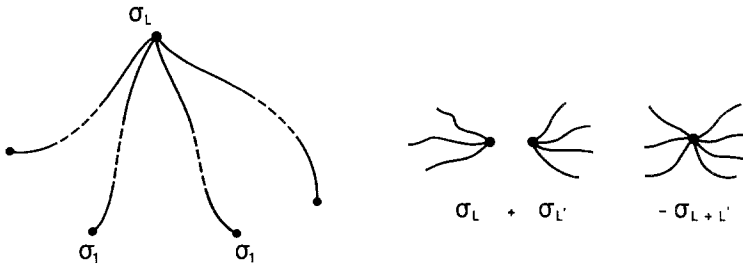


Fig. 2. Decomposition of universal exponents γ_L and $\theta_{L,L'}$ of Fig. 1 into their irreducible vertex parts $\{\sigma_i\}$: $\gamma_L - 1 = \sigma_L + L\sigma_1$, $\nu\theta_{L,L'} = \sigma_L + \sigma_{L'} - \sigma_{L+L'}$.

Note that the des Cloizeaux contact exponents θ_1, θ_2 of a single chain⁽⁹⁾ read in our formalism $\theta_1 = \theta_{L=1, L'=2}$ and $\theta_2 = \theta_{L=2, L'=2}$ (Fig. 3). Hence we find

$$\begin{aligned} \theta_1 &= (2\gamma - \gamma_3 - 1)/\nu \\ \theta_2 &= (2\gamma - \gamma_4 - 1)/\nu \end{aligned} \tag{1.9}$$

relations which were also guessed in ref. 14.

How does all this fit in the usual field-theoretic description of critical phenomena^{(15)?}

The scaling dimensions σ_L correspond⁽¹⁾ actually in $(\varphi^2)^2 O(n)$ -field theory to the anomalous dimensions of successive powers of the field φ^L . Usually, one is interested, in the statistical mechanics of critical points, only in the scaling dimensions of the first powers φ and φ^2 , which lead the usual exponents η and ν ,⁽³⁾ i.e., also to $\gamma = (2 - \eta)\nu$.

But there exist also higher independent scaling dimensions associated with φ^L , leading to new, higher critical exponents, but these operators are not so much considered, perhaps both because they have no immediate physical interpretation, and because their field-theoretic renormalization for high L 's is not simple to perform. In polymer physics (which corresponds to the particular case $n \rightarrow 0$) the situation is quite different. These exponents have an immediate geometrical meaning, and furthermore they are relatively easily accessible by the technique of *direct* renormalization, specially tailored for polymers.⁽⁵⁾

Let us now return to polymers. From the basic exponents σ_L , one can now construct the scaling theory of any polymer network.⁽¹⁾

Consider indeed a branched polymer \mathcal{G} (Fig. 4) of arbitrary but fixed topology, made of \mathcal{N} chains of equal lengths N , tied together at vertices of

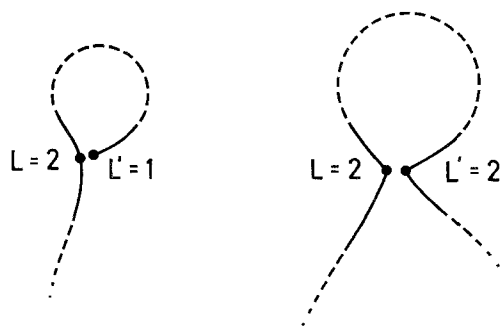


Fig. 3. Usual contact exponents θ_1, θ_2 inside a chain. They depend only on the vertices upon contact and not on the connectivity of the chain. They are star contact exponents $\theta_{L,L'}$ (see Fig. 1), namely those of 1-leg and 2-leg stars: $\theta_1 \equiv \theta_{1,2}$; and of two 2-leg stars: $\theta_2 \equiv \theta_{2,2}$.

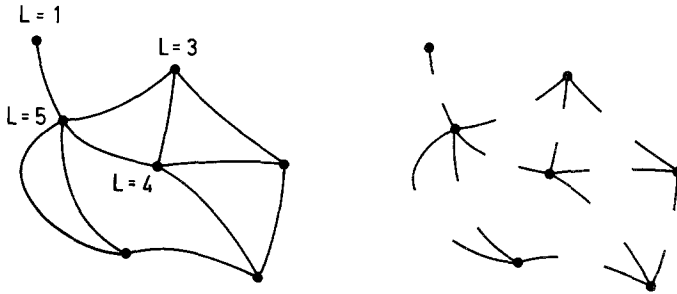


Fig. 4. A polymer network \mathcal{G} of arbitrary but fixed topology, specified by the numbers n_L of L -leg vertices inside \mathcal{G} : Here $n_1 = 1, n_3 = 4, n_4 = 1, n_5 = 1$. The configuration exponent $\gamma_{\mathcal{G}}$ is the sum of all the contributions σ_L of the physical L -leg vertices $\gamma_{\mathcal{G}} - 1 = -vd\mathcal{L} + \sum_{L \geq 1} n_L \sigma_L$, where $\mathcal{L} = 1 + \frac{1}{2} \sum_L n_L (L - 2)$ is the Euler number of loops of graph \mathcal{G} . It can thus also be written as $\gamma_{\mathcal{G}} - 1 = -vd + \sum_{L \geq 1} n_L A_L$ with $A_L = \sigma_L - vd(L - 2)/2$.

functionalities L ($L \geq 1$) and in number n_L , and ask about the asymptotic number of self-avoiding configurations $\mathcal{Z}_{\mathcal{G}}$ of \mathcal{G} . The answer is⁽¹⁾

$$\mathcal{Z}_{\mathcal{G}} \sim \mu^{\mathcal{N}} N^{\gamma_{\mathcal{G}} - 1} \quad (N \gg 1) \tag{1.10}$$

where $\gamma_{\mathcal{G}}$ is the topology-dependent critical exponent:

$$\gamma_{\mathcal{G}} - 1 = -v d\mathcal{L} + \sum_{L \geq 1} n_L \sigma_L \tag{1.11}$$

where \mathcal{L} is the number of physical loops in the polymer network \mathcal{G} . In (1.11) one observes the fundamental result that each L -vertex contributes by its scaling dimension σ_L as many times n_L it appears in \mathcal{G} . Note also that $L = 1$ corresponds to the free extremities of the network. A necessary requirement of the theory is that⁽¹⁾

$$\sigma_2 \equiv 0 \tag{1.12}$$

since $L = 2$ corresponds to the two-leg vertices, i.e., to any “monomer” or any point in the network, which forms a continuum set, hence $n_2 = \infty$. We shall see that this is indeed always the case.

All the fundamental polymer scaling theory described here applies in any dimension d . Above four dimensions ($d > 4$) the self-avoidance effects are irrelevant. Hence all critical exponents are trivial, and we have simply

$$\sigma_L = 0, \quad \forall L \geq 1, \quad d > 4 \tag{1.13}$$

When $d \leq 4$, $d = 4 - \varepsilon$, the σ_L can be calculated analytically via the ε -expansion^(1,5)

$$\sigma_L = (2 - L)L\varepsilon/16 + O(\varepsilon^2) \tag{1.14}$$

The theory applies equally well for self-avoiding walks in *two dimensions*. We know there the infinite set of exact values

$$\sigma_L = (2 - L)(9L + 2)/64, \quad L \geq 1 \quad (1.15)$$

derived⁽¹⁾ from conformal invariance⁽¹²⁾ or Coulomb-gas^(16,17) results.

Note also that in two dimensions a very interesting case is that of *dense* polymers,^(13,18-21) which fill a finite fraction of the (infinite) lattice. They correspond to the critical low-temperature phase of the $O(n)$ model^(13,18,19) for $n \rightarrow 0$, or to the Q -state Potts model at its critical point for $Q \rightarrow 0$.^(20,21) Then the fundamental σ_L exponents read^(13,18)

$$\sigma_L = -(L^2 - 4)/32 \quad (L \geq 1) \quad (1.16)$$

The aim of the present paper is twofold. First, to explain the roots of the decomposition in terms of vertex operators for any scaling function for polymers. In other words, I want to unify and derive in a simple way the results (1.4)–(1.11) in terms of the vertex exponents σ_L , and this paper can be seen as the detailed expansion of ref. 1. A second aim is to calculate these vertex exponents σ_L in second order in ε . One finds

$$\sigma_L = (\varepsilon/8)(2 - L)L/2 + (\varepsilon/8)^2 L(L - 2)(8L - 21)/8 + O(\varepsilon^3) \quad (1.17)$$

This leads to the practical knowledge of an infinite series of geometrical exponents for polymers to order $O(\varepsilon^2)$.

I shall also give the values to $O(\varepsilon)$ of the σ_L^S for *surface* critical phenomena in the case of (branched) polymers near a strongly repelling interface^(22,23) (i.e., the ordinary surface transition). The same vertex theory applies also to the case of tricritical polymers at the Θ -point.⁽²⁴⁻²⁶⁾ I give here the analogous vertex dimensions σ_L^Θ to first order in $\varepsilon = 3 - d$, below the upper tricritical dimension $d = 3$.

Note that the idea of introducing exponents for the vertices higher than 2 appeared in ref. 5 but was not exploited there. Des Cloizeaux and Jannink⁽⁵⁾ used it to calculate contact exponents to $O(\varepsilon)$. In two dimensions, it was studied numerically in ref. 12 for some peculiar “watermelon” topology⁽¹⁾ and by the Coulomb-gas method.^(16,17) More generally, the formalism I shall describe here applies integrally in dimension 2, where it is reinforced by powerful conformal invariance theory⁽²⁷⁾ and Coulomb-gas techniques.⁽¹⁶⁾ See refs. 1, 12, 13, 16–22, 26, and 27 for polymers in two dimensions. There all the geometrical exponents are known exactly.

Note finally that the study I describe here for the vertex theory of topological polymers, i.e., dilute self-avoiding walks, is very similar to the scaling theory of the intersections of random walks given in a previous

study.⁽²⁸⁾ There also, the exponents σ_L corresponding to the noncrossing probability of L independent random walks were calculated to order $O(\varepsilon^2)$.

This paper is organized as follows.

In Section 2, I describe the formalism needed for calculating the partition function of any polymer network, in the framework of the standard Edwards continuum model for polymers in d dimensions. The perturbative rules are given. The relationship to correlation functions in the field theory or $O(n)$ model (for $n=0$) is explained in the simple and basic case of the watermelon topology (L polymer chains tied together at their extremities). This case is also used to illustrate the (infrared) divergences occurring in perturbation theory.

In Section 3, the renormalization principles for treating any network topology are given. I work in the good solvent case [i.e., near the critical point of the $O(n=0)$ model], but the principles can be extended to any multicritical point for polymers (Section 7 and ref. 28). It is shown that the polymer network partition functions are renormalized by factorization of the critical divergences over the physical vertices of the network.⁽¹⁾ As a consequence, critical exponents for $d < 4$ are sums of contributions of the vertices as in (1.11). A general scaling theory of networks is thus established. The logarithmic behavior in $d=4$ is also given.

In Section 4, the same scaling theory of polymer networks, derived in Section 3 from the direct renormalization method for polymers,⁽⁵⁾ is now rederived in field theory, by considering fully polydisperse networks. This scaling method was introduced in ref. 22 together with conformal invariance studies. The polydisperse network partition functions appear in the field theory as correlators of composite operators ϕ_L associated with the L -leg vertices.

In Section 5 the basic vertex exponents σ_L , (1.17), of the L -leg vertices or the associated scaling dimensions x_L of the field operators ϕ_L are calculated to $O(\varepsilon^2)$ in $d=4-\varepsilon$ dimensions. For this, I use the direct renormalization method and calculate *directly* the star partition functions (1.4) and their critical exponent γ_L . All other exponents follow, such as the contact exponents (1.8). I also illustrate the scaling theory by giving all the corresponding values in two dimensions, which are exactly known.

Section 6 introduces the new surface effects for polymers in semi-infinite geometry. The networks are grafted to a Dirichlet surface (ordinary surface transition). I first describe the Brownian case where phantom networks approach the surface. In $d < 4$, in a good solvent, new anomalous dimensions σ_L^S appear for each L -leg vertex grafted onto the surface. I give the associated direct surface renormalization theory for any network. The configuration exponent⁽²²⁾ $\gamma_{\mathcal{G}}^S$ of any network near the surface is then given as a sum over the contributions σ_L of bulk vertices and σ_L^S of surface ones,

generalizing (1.11). Infinities of exact scaling relations between various network surface exponents can be derived. In particular, I give a generalization of Barber's scaling law to L -leg vertices. The set of basic surface exponents σ_L^S , $L \geq 1$, is calculated to $O(\varepsilon)$ in $d = 4 - \varepsilon$, by a direct evaluation of the star surface partition functions. All other surface exponents can be derived from them. The corresponding exact values in 2D are given. In $d = 4$ the surface logarithmic corrections are also calculated.

In Section 7, the above scaling theory is generalized to polymer networks in a Θ -solvent (tricritical case), as well as its direct renormalization. At the upper tricritical dimension $d = 3$, the logarithmic correction is thus calculated for an arbitrary polymer network. For instance, the partition function of an L -arm star scales at the Θ -point exactly as

$$\mathcal{Z}(\mathcal{G}_L) \sim (\ln N)^{-L(L-1)(L-2)/132}$$

yielding new logarithmic effects for $L \geq 3$.

Finally, in Section 8, the predictions of the hyperscaling theory of polymer networks are compared to existing numerical simulations. The best test is given by the 2D case, where all theoretical results are expected to be exact. The agreement with the numerical estimates gathered through the literature for networks such as stars, H -comb, etc., in the bulk, or near surfaces and wedges, is excellent.

2. CONTINUUM MODEL OF POLYMER NETWORKS

2.1. Partition Functions

We start by considering \mathcal{N} independent linear polymer chains $a = 1, \dots, \mathcal{N}$, tied together at some vertices and forming a connected network⁽¹⁾ \mathcal{G} . The positions in space \mathbb{R}^d of the vertices are not fixed, but the *topology* of \mathcal{G} , though arbitrary, is fixed. It is characterized by the set of numbers of times n_L the vertices of L legs appear in \mathcal{G} :

$$n_L = \# \text{ of } L\text{-vertices of } \mathcal{G}, \quad L \geq 1$$

For $L = 1$, n_1 is the number of free ends of \mathcal{G} floating in the solvent. Of course, the set $\{n_L\}$ does not completely characterize the topology of \mathcal{G} : two different networks may have the same sets of vertex numbers $\{n_L\}$. However, this set is sufficient for describing the critical properties of \mathcal{G} . The probability weight describing the configurations of the \mathcal{N} chains is taken as the standard Edwards' continuum action⁽²⁹⁾

$$\begin{aligned}
 \mathcal{P}_{\mathcal{N}}\{\mathbf{r}_a\} &= \exp(-\mathcal{A}\{\mathbf{r}_a\}) \\
 \mathcal{A}\{\mathbf{r}_a\} &= \frac{1}{2} \sum_{a=1}^{\mathcal{N}} \int_0^S \left(\frac{d\mathbf{r}_a(s)}{ds} \right)^2 ds \\
 &\quad + \frac{1}{2} b \sum_{a=1}^{\mathcal{N}} \sum_{a'=1}^{\mathcal{N}} \int_0^S ds \int_0^S ds' \delta^d(\mathbf{r}_a(s) - \mathbf{r}_{a'}(s'))
 \end{aligned} \tag{2.1}$$

where $\mathbf{r}_a(s)$ is the configuration of the a chain in space, and where $b (>0)$ is the excluded volume coefficient. Now, the partition function of the connected network \mathcal{G} reads formally as a functional integral

$$\mathcal{Z}(\mathcal{G}) = \int \prod_{a=1}^{\mathcal{N}} d\{\mathbf{r}_a\} \delta^d[\mathcal{G}] \mathcal{P}_{\mathcal{N}}\{\mathbf{r}_a\} \left/ \left[\int d\{\mathbf{r}\} P_0\{\mathbf{r}\} \delta^d(\mathbf{r}(0)) \right]^{\mathcal{N}} \right. \tag{2.2}$$

where $\delta^d[\mathcal{G}]$ is a symbolic notation for all the δ distributions necessary for building the network \mathcal{G} . In the denominator of (2.2), $P_0\{\mathbf{r}\}$ is the Brownian weight for a single chain, obtained from (2.1) for $\mathcal{N} = 1$ and $b = 0$, hence $\mathcal{Z}(\mathcal{G})$ is normalized by the total number of configurations of \mathcal{N} Brownian paths, with their origin fixed. In this way, $\mathcal{Z}(\mathcal{G})$ admits a *well-defined finite continuum limit*.^(1,5)

The $\delta^d[\mathcal{G}]$ distribution of the connected networks \mathcal{G} reads explicitly

$$\begin{aligned}
 \delta^d[\mathcal{G}] &= \int \prod_{i=1}^{\mathcal{V}-1} d^d R_i \prod_{i=1}^{\mathcal{V}} \\
 &\quad \times \left[\prod_{a \in L^+(i)} \delta^d(\mathbf{r}_a(0) - \mathbf{R}_i) \prod_{a' \in L^-(i)} \delta^d(\mathbf{r}_{a'}(S) - \mathbf{R}_i) \right]
 \end{aligned} \tag{2.3}$$

where $i = 1, \dots, \mathcal{V}$ denotes the set of the \mathcal{V} vertices of \mathcal{G} , and where $L^+(i)$ is the set of oriented polymer lines $\{a\}$ leaving the vertex i , and $L^-(i)$ the set of polymer lines arriving at i (Fig. 4). The natural orientation chosen on each polymer chain is that of its abscissa $s \in [0, S]$. [The distinction $L^+(i)$, $L^-(i)$ is purely a matter of notation since we have to distinguish $\mathbf{r}_a(0)$ and $\mathbf{r}_a(S)$ on each chain]. Note that in (2.3) we integrate over the position of $\mathcal{V} - 1$ vertices only, in order to cancel the translational invariance infinite-volume factor.

2.2. Cutoff and Dimensional Regularizations

It must be realized that interaction integrals like the $b \int \delta^d(\mathbf{r}(s) - \mathbf{r}(s')) ds ds'$ in Eq. (2.1) yield divergent contributions when $s = s'$.^(5,6,30) So one has to regularize the theory and a physically appealing way is to introduce a short-distance ultraviolet cutoff “area”⁽⁵⁾ such that in any

interaction integral $|s - s'| \geq s_0$. Then $\mathcal{Z}(\mathcal{G})$, (2.2), really depends on variables b, S, s_0, d . In the limit $s_0 \rightarrow 0$, and for $d < 4$, it is possible to factorize out the s_0 dependence.⁽⁵⁾ One has

$$\lim_{s_0 \rightarrow 0} \mathcal{Z}(\mathcal{G}, b, S, s_0, d) = \exp[\mathcal{N}(S/s_0) C(z_0)] \mathcal{Z}(\mathcal{G}, b, S, d)|_{\text{dim.reg.}} \quad (2.4)$$

where z_0 is the cutoff-dependent and dimensionless interaction parameter⁽⁵⁾

$$z_0 = (2\pi)^{-d/2} b s_0^{2-d/2}$$

and $C(z_0)$ is a regular function⁽⁵⁾ calculable as a power series of z_0 . $C(z_0)$ is independent of \mathcal{G} and can be seen as a shift in the free energy per monomer, $\mathcal{N}S/s_0$ being the “number” of monomers in our continuous model. On the right-hand side of (2.4), $\mathcal{Z}(\mathcal{G})|_{\text{dim.reg.}}$ is the *dimensionally regularized*⁽³¹⁾ partition function, obtained by *continuing analytically* the polymer model, free of ultraviolet divergences for $d < 2$, to values $d > 2$. This partition function will depend on S and on the dimensionless interaction parameter^(5,32)

$$z = (2\pi)^{-d/2} b S^{2-d/2} \quad (2.5)$$

and is the mathematically attractive object. Note that in our cutoff continuum model the *effective connectivity constant* μ of (1.10) is simply $\mu \equiv e^{C(z_0)}$. Hence we expect the partition function in dimensional regularization to scale like

$$\mathcal{Z}(\mathcal{G}, b, S, d)|_{\text{dim.reg.}} \sim S^{\gamma_{\mathcal{G}}-1} \quad (S \rightarrow \infty) \quad (2.6)$$

i.e., in dim.reg. $\mu \equiv 1$. From now on we work *exclusively in dimensional regularization*.

2.3. Brownian Partition Function $\mathcal{Z}^B(\mathcal{G})$

To investigate the structure of $\mathcal{Z}(\mathcal{G})$, we first need to perform its naive dimensional analysis. The number of *effective* δ interactions in (2.3) is

$$A = \sum_{i=1}^{\mathcal{V}} L(i) - (\mathcal{V} - 1) \quad (2.7)$$

where $L(i) = L^+(i) + L^-(i)$ is the total number of polymer lines attached at vertex i . The subtraction of $\mathcal{V} - 1$ in (2.7) amounts to the integration over the $\mathcal{V} - 1$ positions of the vertices. For the polymer network \mathcal{G} described by the set $\{n_L\}$ of numbers of vertices of type L we can rewrite (2.7) as

$$A = \sum_{L \geq 1} n_L(L - 1) + 1 \quad (2.8)$$

since trivially

$$\mathcal{V} = \sum_{L \geq 1} n_L \quad (2.9)$$

The canonical dimension of $\delta^d[\mathcal{G}]$, (2.3), is then

$$\delta^d[\mathcal{G}] \sim S^{-d\mathcal{A}/2} \quad (2.10)$$

where S is the Brownian *area*⁽⁵⁾ of the chain, homogeneous to a length squared [see (2.1)]. Finally, the normalized partition function $\mathcal{Z}(\mathcal{G})$, (2.2), scales necessarily like (in dim.reg.)

$$\mathcal{Z}(\mathcal{G}) = (2\pi S)^{(\mathcal{N} - \mathcal{A})d/2} Z(\mathcal{G}, z, d) \quad (2.11)$$

where $Z(\mathcal{G}, z, d)$ is a *dimensionless* function of the dimensionless interaction parameter z .

It will be useful to know the topological relations in a network \mathcal{G} made of \mathcal{N} chains,

$$2\mathcal{N} = \sum_{L \geq 1} L n_L \quad (2.12)$$

$$\mathcal{L} = \sum_{L \geq 1} \frac{1}{2} (L - 2) n_L + 1 \quad (2.13)$$

where \mathcal{L} is the Euler number of physically independent *loops* in \mathcal{G} (Fig. 4). With these, the canonical dimension of $\mathcal{Z}(\mathcal{G})$, (2.11), is, in S units,

$$\mathcal{D} = (\mathcal{N} - \mathcal{A})d/2 = -\mathcal{L}d/2$$

and

$$\mathcal{Z}(\mathcal{G}) = (2\pi S)^{-\mathcal{L}d/2} Z(\mathcal{G}, z, d) \quad (2.14)$$

Note that for a pure Brownian network ($b = 0$) the functional integral (2.2) can be performed exactly as^(21,28)

$$\mathcal{Z}^B(\mathcal{G}) = (2\pi S)^{-\mathcal{L}d/2} (\det \mathbb{C}_{\mathcal{G}/\{a\}})^{-d/2} \quad (2.15)$$

where $\mathbb{C}_{\mathcal{G}}$ is the *connection matrix*⁽²¹⁾ of the graph \mathcal{G} . It is a $\mathcal{V} \times \mathcal{V}$ matrix, whose elements \mathbb{C}_{ab} are labeled by the vertices of \mathcal{G} and are defined as

$$\begin{aligned} \text{if } a \neq b & \quad \mathbb{C}_{ab} = -\# \text{ lines joining } a \text{ to } b \\ \text{if } a = b & \quad \mathbb{C}_{aa} = - \sum_{b \neq a} \mathbb{C}_{ab} \\ & = \# \text{ lines attached to } a \text{ by one} \\ & \quad \text{(and only one) extremity} \end{aligned} \quad (2.16)$$

This matrix has a zero mode,⁽²¹⁾ and hence $\det \mathbb{C}_{\mathcal{G}} \equiv 0$. In (2.15), $\det \mathbb{C}_{\mathcal{G}/\{a\}}$ is then the minor of $\mathbb{C}_{\mathcal{G}}$ with respect to any vertex a of \mathcal{G} (which is independent of a).

Formula (2.15) hence fully confirms the result (2.14) of dimensional analysis. We can furthermore make precise the Brownian normalization of Z for $z = 0$:

$$Z^B(\mathcal{G}, d) \equiv Z(\mathcal{G}, z = 0, d) = (\det \mathbb{C}_{\mathcal{G}/\{a\}})^{-d/2} \tag{2.17}$$

Now the problem has been reduced to the analysis of dimensionless $Z(\mathcal{G}, z, d)$. Its scaling behavior will be found as usual in direct polymer theory^(5,1) by perturbation expansion followed by renormalization. Now, the dimensionally regularized partition function $Z(\mathcal{G}, z, d)$, when expanded in powers of z , develops poles at some values of the dimension d , for $2 \leq d \leq 4$. These polar dimensions are well known in the $(\varphi^2)^2$ field theory⁽³¹⁾ or in polymer theory⁽³³⁾:

$$d(p) = 4 - 2/p, \quad p \in \mathbb{N}^*$$

They include $d = 2$, $d = 3$, and finally $d(\infty) = 4$, which is the upper critical dimension of the theory. So we set $d = 4 - \varepsilon$, and $Z(\mathcal{G}, z, 4 - \varepsilon)$ has the double Laurent–Taylor series expansion

$$\frac{Z(\mathcal{G}, z, d)}{Z^B(\mathcal{G}, d)} = 1 + \sum_{n \geq 1} a_n(\varepsilon) z^n, \quad d = 4 - \varepsilon \tag{2.18}$$

$$a_n(\varepsilon) = \frac{\alpha_{n,n}}{\varepsilon^n} + \frac{\alpha_{n,n-1}}{\varepsilon^{n-1}} + \dots + \frac{\alpha_{n,1}}{\varepsilon} + \alpha_{n,0}(\varepsilon)$$

where $\alpha_{n,0}$ is regular when $\varepsilon \rightarrow 0$. The coefficients $a_n(\varepsilon)$ are essential^(34,35) and are found by perturbation expansion.

2.4. Perturbation Expansion

The perturbative rules for calculating any partition function $\mathcal{Z}(\mathcal{G})$ can be obtained by generalizing those of ref. 5. One expands the weight (2.1) in powers of the interaction b , or equivalently of the dimensionless parameter z , (2.5). Each δ interaction is Fourier transformed into

$$\delta^d[\mathbf{r}_a(s) - \mathbf{r}_a(s')] = (2\pi)^{-d} \int d^d q \exp\{i\mathbf{q} \cdot [\mathbf{r}_a(s) - \mathbf{r}_a(s')]\} \tag{2.19}$$

and the averages with independent Brownian weights of the expanded terms are easily performed with the simple Green function for each chain a :

$$\langle \exp\{i\mathbf{q} \cdot [\mathbf{r}_a(s) - \mathbf{r}_a(s')]\} \rangle_0 = \exp(-\frac{1}{2}q^2 |s - s'|) \tag{2.20}$$

In our case, the situation is slightly complicated by the fact that there are additional $\delta^d[\mathcal{G}]$ distributions, (2.3), building the connections of the network \mathcal{G} . This manifests itself in the fact that the pure Brownian partition function (2.15) is not trivial. These $\delta^d[\mathcal{G}]$ distributions are also Fourier transformed as in (2.19). The diagrammatic rules for calculating $\mathcal{Z}(\mathcal{G})$ are then the following.

1. Diagrams are made of \mathcal{N} continuous chains building the network \mathcal{G} at some prescribed L -leg vertices. This “bare” topology of \mathcal{G} is *conserved* for all diagrams of the perturbation expansion of $\mathcal{Z}(\mathcal{G})$ in powers of b . Then *dotted lines* joining two *interacting* points on the various chains of the network are introduced, with a factor $-b$ for each interaction.

2. A set of independent *loops* is selected in each interaction diagram, which involve continuous polymer chains and interaction dotted lines. Some of these loops can be made of polymers lines only and are thus constitutive loops of the “bare” Gaussian network \mathcal{G} itself. An independent momentum \mathbf{q} flows along each loop, with integration measure

$$\int \frac{d^d q}{(2\pi)^d}$$

[see (2.19)].

3. Along each segment of length s of a chain, determined by two successive interaction points along the *same* chain, one evaluates the total momentum flowing along it, which is the algebraic sum $\sum \mathbf{q}$ of the momenta of all the independent loops to which the segment belongs. This segment contributes then a factor $\exp[-\frac{1}{2}(\sum \mathbf{q})^2 s]$ to the integrand.

4. One integrates over all independent momenta, and all positions of interaction points which preserve the topology of interactions of the diagram.

5. One sums over all possible topologies of the interaction lines, keeping the constitutive topology of \mathcal{G} itself fixed.

I give an example on a nontrivial diagram which will be useful later.

2.5. Watermelon Network

Consider a graph \mathcal{G} (“watermelon network”) made of L polymer lines of equal length S tied together at their extremities \mathbf{X} and \mathbf{Y} (Fig. 5). This kind of configuration plays a central role⁽¹⁾ in two dimensions, where it can be analyzed^(12,1,13,16,17,20) by Coulomb-gas and conformal invariance methods. When the lengths of the chains fluctuate, it corresponds to

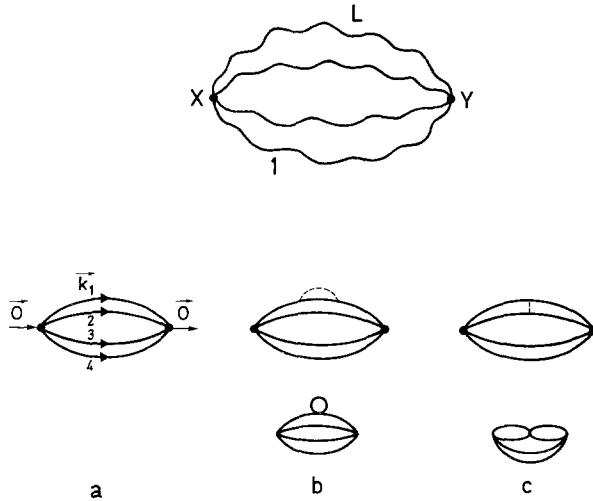


Fig. 5. The watermelon network \mathcal{W}_L of L polymer lines tied together, and the first Feynman diagrams contributing to its partition function $\mathcal{Z}(\mathcal{W}_L)$ [Eqs. (2.24), (2.26), (2.27)].

multispin (totally connected) correlation functions in the associated $O(n)$ model ($n \rightarrow 0$).

2.5.1. Partition Function. So let us for a while consider a polydisperse network and take L chains of lengths S_1, \dots, S_L . The diagrams contributing to $\mathcal{Z}_L(S_1, \dots, S_L, b, d)$ to first order in excluded volume b are given in Fig. 5.

First the Brownian value \mathcal{Z}^B can be calculated via the above rules. It reads in Fourier representation

$$\mathcal{Z}_L^B(S_1, \dots, S_L, d) = \int \prod_{i=1}^L \frac{d^d k_i}{(2\pi)^d} \times (2\pi)^d \delta^d\left(\sum_{i=1}^L \mathbf{k}_i\right) \prod_{i=1}^L \exp\left(-\frac{1}{2} k_i^2 S_i\right)$$

We rewrite it in direct space as

$$\mathcal{Z}_L^B\{S_i\} = \int d^d X \prod_{i=1}^L G(\mathbf{X}, S_i) \tag{2.21}$$

where

$$\begin{aligned} G(\mathbf{X}, S) &\equiv \int \frac{d^d k}{(2\pi)^d} \exp(-k^2 S/2 - i\mathbf{k} \cdot \mathbf{X}) \\ &= (2\pi S)^{-d/2} \exp(-X^2/2S) \end{aligned} \tag{2.22}$$

Note that $G(\mathbf{X}, S)$ is simply the Brownian probability distribution

$$G(\mathbf{X}, S) = \langle \delta^d(\mathbf{r}(S) - \mathbf{r}(0) - \mathbf{X}) \rangle_0 \tag{2.23}$$

So we find for (2.21)

$$\begin{aligned} \mathcal{Z}_L^B\{S_i\} &= \prod_{i=1}^L (2\pi S_i)^{-d/2} \int d^d X \exp\left(-\frac{1}{2} X^2 \sum_{i=1}^L S_i^{-1}\right) \\ &= \prod_{i=1}^L (2\pi S_i)^{-d/2} (2\pi)^{d/2} \left(\sum_{i=1}^L \frac{1}{S_i}\right)^{-d/2} \end{aligned} \tag{2.24}$$

In the monodisperse case it reads

$$\mathcal{Z}_L^B(L \times S) = (2\pi S)^{-(L-1)d/2} L^{-d/2} \tag{2.25}$$

This is in agreement with the general formula (2.15), since here the connectivity matrix is simply

$$\mathbb{C} = \begin{pmatrix} L & -L \\ -L & L \end{pmatrix}$$

and its minors with respect to the two vertices are L .

Now let us consider the two interaction diagrams of Figs. 5b and 5c, with contributions

$$\begin{aligned} \mathcal{J}_1 &= -b \int \prod_{i=1}^L \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta^d\left(\sum_{i=1}^L k_i\right) \prod_{i=2}^L \exp\left(-\frac{1}{2} k_i^2 S_i\right) \\ &\quad \times \int_0^{S_1} ds (S_1 - s) \exp\left[-\frac{1}{2} k_1^2 (S_1 - s)\right] \int \frac{d^d k}{(2\pi)^d} \exp\left(-\frac{1}{2} k^2 s\right) \\ \mathcal{J}_2 &= -b \int \prod_{i=1}^L \frac{d^d k_i}{(2\pi)^d} (2\pi)^d \delta^d\left(\sum_{i=1}^L k_i\right) \prod_{i=3}^L \exp\left(-\frac{1}{2} k_i^2 S_i\right) \\ &\quad \times \int \frac{d^d k}{(2\pi)^d} \int_0^{S_1} ds_1 \int_0^{S_2} ds_2 \exp\left[-\frac{1}{2} k_1^2 (S_1 - s_1) - \frac{1}{2} (\mathbf{k}_1 + \mathbf{k})^2 s_1\right] \\ &\quad \times \exp\left[-\frac{1}{2} k_2^2 (S_2 - s_2) - \frac{1}{2} (\mathbf{k}_2 - \mathbf{k})^2 s_2\right] \end{aligned}$$

The momentum integrations are easy to perform via Gaussian integration formulas. One finds

$$\begin{aligned} \mathcal{J}_1 &= -b(2\pi)^{-d/2} \prod_{i=2}^L (2\pi S_i)^{-d/2} \\ &\quad \times \int_0^{S_1} ds (S_1 - s)^{1-d/2} s^{-d/2} \left(\frac{1}{S_1 - s} + \sum_{j=2}^L \frac{1}{S_j}\right)^{-d/2} \end{aligned} \tag{2.26}$$

and

$$\mathcal{J}_2 = -b(2\pi)^{-d/2} \int_0^S ds_1 \int_0^S ds_2 (s_1 + s_2)^{-d/2} (2\pi)^{-(L-1)d/2} \prod_{j \geq 3} (S_j)^{-d/2} \times \left[S_1 + S_2 - s_1 - s_2 + \left(\sum_{j \geq 3} \frac{1}{S_j} \right) \left(S_1 S_2 - \frac{S_1 s_2^2 + S_2 s_1^2}{s_1 + s_2} \right) \right]^{-d/2} \quad (2.27)$$

I have given these formulas to show on a specific example the kind of perturbation theory and Feynman–Schwinger type integrals one can expect for these interacting networks with nontrivial topology. Of course, for the L -chains of the peculiar watermelon network \mathcal{W}_L one has to add all the contributions obtained from (2.26) and (2.27) by permuting S_1 with the S_j and S_1, S_2 with any pair (S_i, S_j) :

$$\mathcal{Z}_L\{S_j\} = \mathcal{Z}_L^B\{S_j\} + \sum_i \mathcal{J}_1(S_i) + \sum_{\substack{i,j \\ i \neq j}} \mathcal{J}_2(S_i, S_j) \quad (2.28)$$

Let us now be more specific and consider the *monodisperse* case where all the chains have the same length:

$$S_j = S \quad \forall j = 1, \dots, L \quad (2.29)$$

Then we find the perturbative expansion

$$\mathcal{J}_1 = -z(2\pi S)^{-(L-1)d/2} I_d \quad (2.30)$$

$$I_d = \int_0^1 [1 + (L-1)(1-x)]^{-d/2} (1-x) x^{-d/2} dx \quad (2.30a)$$

$$z = (2\pi)^{-d/2} b S^{2-d/2}$$

and

$$\mathcal{J}_2 = -z(2\pi S)^{-(L-1)d/2} I'_d \quad (2.31)$$

$$I'_d = \int_0^1 dx_1 \int_0^1 dx_2 [(x_1 + x_2)L - (x_1 + x_2)^2 - (L-2)(x_1^2 + x_2^2)]^{-d/2} \quad (2.31a)$$

To first order in b or z , the perturbation expansion of the partition function of the watermelon network \mathcal{W}_L is thus

$$\begin{aligned} \mathcal{Z}(\mathcal{W}_L) &\equiv \mathcal{Z}_L(L \times S) = (2\pi S)^{-d(L-1)/2} L^{-d/2} + L \mathcal{J}_1 + \frac{1}{2} L(L-1) \mathcal{J}_2 \\ &= (2\pi S)^{-d(L-1)/2} [L^{-d/2} - z L I_d - z \frac{1}{2} L(L-1) I'_d] \end{aligned}$$

and this illustrates perfectly the structure announced in (2.14), (2.15). To see the occurrence of divergences, one has to perform the Laurent expansion of integrals I_d [(2.30a)] and I'_d [(2.31a)] for $d=4-\varepsilon$ near $\varepsilon=0$. The dominant polar $1/\varepsilon$ terms are easy to get. Since later we shall calculate *universal amplitude ratios*, we shall need the subdominant terms. They require some algebra. We finally find the contributions of the diagrams of Figs. 5b and 5c:

$$I_d = L^{-d/2} \frac{1}{L} \left[\frac{2}{\varepsilon} (L-2) + 2 - 3L + (L-2) \ln L \right] \quad (2.32)$$

$$I'_d = L^{-d/2} 2 \left(\frac{2}{\varepsilon} + 1 + \ln L \right) \quad (2.33)$$

and

$$\begin{aligned} \mathcal{Z}(\mathcal{W}_L) &= (2\pi S)^{-d(L-1)/2} L^{-d/2} \\ &\times \left\{ 1 - z \left[\frac{2}{\varepsilon} (L^2 - 2) + 2 - 4L + L^2 + (L^2 - 2) \ln L \right] + O(z^2) \right\} \end{aligned} \quad (2.34)$$

Recall that these expressions are *dimensionally regularized* ones, and (2.34) illustrates the general polar structure mentioned in (2.18).

2.6. Correlation Functions

It is interesting to analyze the correlation function of the extremities of a self-avoiding watermelon network \mathcal{W}_L . For this, I reintroduce fluctuating lengths S_1, \dots, S_L for the L polymer lines of the watermelon, and define their restricted partition function

$$\begin{aligned} \mathcal{Z}_L(S_1, \dots, S_L, \mathbf{X}, \mathbf{Y}) &= \int d\{\mathbf{r}_a\} \prod_{a=1}^L \delta^d(\mathbf{r}_a(0) - \mathbf{X}) \delta^d(\mathbf{r}_a(S) - \mathbf{Y}) \mathcal{P}_L\{\mathbf{r}_a\} \\ &\times \left(\int d\{\mathbf{r}\} P_0\{\mathbf{r}\} \delta^d[\mathbf{r}(0)] \right)^{-L} \end{aligned} \quad (2.35)$$

where the extremities of the network are at \mathbf{X} and \mathbf{Y} . Let us go to a “magnetic” correlation function in the $O(n)$ model ($n \rightarrow 0$) by summing over the fluctuating lengths and define

$$G_L(\mathbf{X} - \mathbf{Y}, T) \equiv \int_0^\infty \prod_{a=1}^L dS_a e^{-T(S_1 + \dots + S_L)} \mathcal{Z}_L(S_1, \dots, S_L, \mathbf{X}, \mathbf{Y}) \quad (2.36)$$

where T is the “temperature” of the $O(n)$ model. We can invert this Laplace transform and get the polydisperse partition function

$$\int_0^\infty dS_1 \cdots dS_L \delta[S - (S_1 + \cdots + S_L)] \mathcal{Z}_L(S_1, \dots, S_L; \mathbf{X}, \mathbf{Y}) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dT e^{TS} G_L(\mathbf{X} - \mathbf{Y}, T) \quad (2.37)$$

where the contour of integration lies on the right-hand side of the singularities of the integrand. The total polydisperse partition function is defined as

$$\mathcal{Z}_L^{\text{poly}}(S) \equiv \int_0^\infty \prod_{a=1}^L dS_a \delta\left(S - \sum_{a=1}^L S_a\right) \mathcal{Z}_L(\{S_a\}) = \frac{1}{2\pi i} \int dT e^{TS} \int d^d Y G_L(\mathbf{X} - \mathbf{Y}, T) \quad (2.38)$$

The *monodisperse* partition function of the watermelon network is related⁽¹⁾ to the polydisperse one by the simple dimensional analysis of the first line of Eq. (2.38),

$$\mathcal{Z}(\mathcal{W}_L) \equiv \mathcal{Z}_L(S, \dots, S) \sim \mathcal{Z}_L^{\text{poly}}(S) S^{-(L-1)} \quad (2.39)$$

Now, the multiple correlation function $G_L(\mathbf{X} - \mathbf{Y}, T)$ in the $O(n)$ model ($n \rightarrow 0$) is a multiple spin correlation function,^(12,13,16,20) but where only *totally connected* diagrams appear. The $O(n)$ model has a critical point at $T = T_c$ such that $\mu = e^{T_c}$ is the effective connectivity constant of the SAW for $n \rightarrow 0$, and in dimensional regularization $\mu = 1$, $T_c = 0$. At the critical point T_c , the mean length $\langle \sum_a S_a \rangle$ is infinite. There one expects the critical correlation function G_L of the L lines to decay algebraically as

$$G_L(\mathbf{X} - \mathbf{Y}, T_c) \sim |\mathbf{X} - \mathbf{Y}|^{-L(d-2) - 2x'_L} \quad (2.40)$$

The exponent $L(d-2)$ is the *Brownian* value of the correlation exponent, as can be seen from Eqs. (2.21) and (2.22), while x'_L is the anomalous part generated by self-avoidance. It is also convenient to define the total scaling dimension x_L of an L -vertex such that

$$G_L(\mathbf{X} - \mathbf{Y}, T_c) = |\mathbf{X} - \mathbf{Y}|^{-2x_L} \quad (2.40a)$$

with

$$x_L \equiv \frac{L}{2}(d-2) + x'_L \quad (2.40b)$$

Note that in two dimensions

$$x_L \equiv x'_L \quad (2D) \quad (2.40c)$$

For the “phantom” Brownian watermelon network, we have

$$\mathcal{Z}_L^B(\{S_i\}, \mathbf{X}, \mathbf{Y}) = \prod_{i=1}^L \{(2\pi S_i)^{-d/2} \exp[-(\mathbf{X} - \mathbf{Y})^2/2S_i]\} \quad (2.41)$$

For the free chains (free field theory) the critical temperature is $T_c = 0$. Hence the Brownian critical correlation function (2.36) factorizes by integrating (2.41) into

$$\begin{aligned} G_L^B(\mathbf{X} - \mathbf{Y}, T_c = 0) &= \left\{ \int_0^\infty dS (2\pi S)^{-d/2} \exp[-(\mathbf{X} - \mathbf{Y})^2/2S] \right\}^L \\ &= [(2\pi)^{-d/2} 2^{d/2-1} \Gamma(d/2 - 1)]^L |\mathbf{X} - \mathbf{Y}|^{-L(d-2)} \end{aligned} \quad (2.42)$$

which is the result (2.40) for $x'_L = 0$.

When the watermelon network is self-avoiding, the critical correlation function (2.40) develops an anomalous dimension x_L depending on the number of lines L of the network. Note that for $L = 1$, G_1 is just the usual spin-spin correlation function of the associated $O(n)$ model and thus

$$x'_1 = \eta/2$$

where η is the usual magnetic exponent. When we are away from the critical point, it is very natural to assume a further scaling behavior of G_L in terms of the ratio $|\mathbf{X} - \mathbf{Y}|/\xi$, where $\xi = |T - T_c|^{-\nu}$ is the correlation length. Hence we set⁽¹³⁾

$$\begin{aligned} G_L(\mathbf{X} - \mathbf{Y}, T) &= |\mathbf{X} - \mathbf{Y}|^{-L(d-2) - 2x'_L} F_L(|\mathbf{X} - \mathbf{Y}| |T - T_c|^\nu) \\ &= |\mathbf{X} - \mathbf{Y}|^{-2x_L} F_L \end{aligned} \quad (2.43)$$

where F_L is some universal function, with $F_L(0)$ finite. This will be used later.

2.7. A Remark on Topological Constraints

It is important to note that we do not consider here *entanglement* problems.⁽³⁶⁻⁴¹⁾ Specific topological constraints arise due to the impossibility for real polymer chains to cross each other or themselves. For instance, for a $L = 3$ watermelon network, various knotted or nonknotted configurations are possible in three dimensions (Fig. 6). In a Gaussian for-

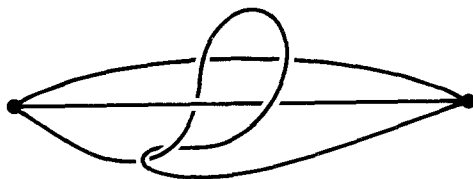


Fig. 6. A knotted configuration of the watermelon. Although it is topologically non-equivalent to the simple configuration of Fig. 5, it is enumerated in $\mathcal{Z}(\mathcal{W}_L)$ for $d=3$.

mula like (2.15) for the Brownian networks, or in perturbative formulas like (2.34), we allow the network to explore all these nonequivalent knot topologies. In this paper only the topology of the branchings is fixed, while that of the knots is not. So, our networks, though self-avoiding in the $b \rightarrow \infty$ limit, are in a sense “phantom” since their Gibbs states fall into several nonequivalent classes with respect to knot topology. It would be quite difficult to consider directly networks in three dimensions, and to implement both excluded volume *and* entanglement specifications (see, however, the results for the rod and single-chain problem in 3D,⁽³⁹⁾ in $d=4-\varepsilon$,⁽⁴⁰⁾ and an exact result in 2D⁽⁴¹⁾).

In some cases, however, the present theory takes into account the proper entanglement constraints: first, when there is only one topological class in the Gibbs space. This is the case for example of the star-polymers. All knotted topologies are accessible by continuous deformations, without cutting the chains. More generally, there are no knotting constraints for any polymer network which has no loop and has a treelike structure. Constraints appear only when the network possesses constitutive loops.

Second, in *two dimensions*,⁽¹⁾ it is interesting to remark that self-avoiding polymer networks are never knotted. Indeed, there are no crossings of the polymer lines and no topological knot constraints appear. Actually, in 2D, a polymer line could be knotted not with another line, but only with points.^(40,41) More generally, the problem of knots for several polymer lines is specific of *three* dimensions. It is not seen in $4-\varepsilon$ dimensions, since the space is too “large.”

3. RENORMALIZATION BY STARS

Consider again the most general polymer network \mathcal{G} and its partition function $\mathcal{Z}(\mathcal{G})$, (2.14). According to (2.6), we expect it to scale with a specific critical exponent $\gamma_{\mathcal{G}}$, universal but topology dependent. Due to (2.18), the perturbation expansion of $\mathcal{Z}(\mathcal{G})$ in powers of z is singular in $\varepsilon=4-d$, and needs to be renormalized. This renormalization will also yield the value of $\gamma_{\mathcal{G}}$. Before giving the principles,⁽¹⁾ we need a reminder of direct renormalization theory for simple linear polymers.⁽⁵⁾

3.1. A Primer in Direct Renormalization^(5,42,43)

Consider simple linear polymer chains described by the continuum Edwards model (2.1). The physical size of an isolated polymer chain is measured by the end-to-end distance

$$R^2 = \langle [\mathbf{r}(S) - \mathbf{r}(0)]^2 \rangle \quad (3.1)$$

Note that the size of a chain inside a network \mathcal{G} is the same, up to some finite amplitude depending on the location along the network. For a Brownian chain one has simply

$$R^2 = dS \quad (3.2)$$

For a continuum polymer chain, described by the Edwards model, R^2 becomes⁽⁵⁾ a function of the dimensionless interaction parameter z , (2.5):

$$R^2 = \mathcal{X}_0(z, d) dS \quad (3.3)$$

where \mathcal{X}_0 is the swelling factor.⁽⁵⁾ Its singular Taylor–Laurent expansion in powers of z and $1/\varepsilon$ reads⁽⁵⁾

$$\mathcal{X}_0(z, d) = 1 + z \left(\frac{2}{\varepsilon} - 1 \right) + z^2 \left(-\frac{6}{\varepsilon^2} + \frac{11}{2\varepsilon} \right) + \dots \quad (3.4)$$

and renormalization amounts to extracting from this singular expansion the asymptotic behavior of $\mathcal{X}_0(z)$, $z \rightarrow \infty$.

In the asymptotic limit $S \rightarrow \infty$, $z \rightarrow \infty$, \mathcal{X}_0 admits the scaling behavior⁽⁵⁾

$$\mathcal{X}_0(z, d) \rightarrow A_0(\varepsilon) z^{(2\nu - 1)2/\varepsilon}, \quad z \rightarrow \infty \quad (3.5)$$

where $A_0(\varepsilon)$ is a calculable amplitude⁽⁴²⁾ and where ν is the correlation length exponent (1.3) for $d = 4 - \varepsilon$ such that $R^2 \sim S^{2\nu}$. We return below to the method for deriving (3.5) from (3.4).

Another important quantity is the single-chain partition function, which in the present formalism is the 1-star partition function $\mathcal{Z}(\mathcal{S}_1, b, S, d)$. It reads, due to (2.14),

$$\mathcal{Z}(\mathcal{S}_1, b, S, d) = Z(\mathcal{S}_1, z, d) \quad (3.6)$$

Its singular perturbation expansion is⁽⁵⁾

$$\mathcal{Z}(\mathcal{S}_1) = 1 + z \left(\frac{2}{\varepsilon} + 1 \right) + z^2 \left(-\frac{6}{\varepsilon^2} - \frac{7}{\varepsilon} \right) + \dots \quad (3.7)$$

and from it, Wilson's method yields the asymptotic behavior^(5,42)

$$\mathcal{Z}(\mathcal{S}_1) \rightarrow A_1(\varepsilon) z^{(\gamma-1)2/\varepsilon}, \quad z \rightarrow \infty \tag{3.8}$$

where γ is the usual magnetic susceptibility exponent (1.3), and $A_1(\varepsilon)$ is calculable.⁽⁴²⁾ The renormalized theory will be expressed in terms of the dimensionless second virial coefficient g ,⁽⁵⁾ such that the osmotic pressure expands like

$$\Pi\beta = \mathbb{C} + \frac{1}{2} g \mathbb{C}^2 (2\pi R^2/d)^{d/2} + \dots \tag{3.9}$$

where \mathbb{C} is the chain concentration. In terms of connected partition functions, g reads

$$g = - \frac{\mathcal{Z}(\mathcal{S}_1 \times \mathcal{S}_1)}{[\mathcal{Z}(\mathcal{S}_1)]^2} \left(\frac{2\pi R^2}{d} \right)^{-d/2} \tag{3.10}$$

where $\mathcal{Z}(\mathcal{S}_1 \times \mathcal{S}_1)$ is the *connected* partition function of two independent chains (see refs. 5 and 43 for precise definitions). It is interesting to note its Taylor–Laurent expansion in powers of z ⁽⁵⁾ and ε :

$$g = z + z^2 \left[-\frac{8}{\varepsilon} + 2 + 4 \ln 2 \right] + z^3 \left[\frac{64}{\varepsilon^2} + \frac{1}{\varepsilon} (-15 - 64 \ln 2) \right] + \dots \tag{3.10bis}$$

The beautiful idea of this direct renormalization theory is that g reaches a finite fixed point value when $z \rightarrow \infty$, and that all scaling functions (in the usual sense of renormalization theory⁽³⁾) have double pure Taylor series expansions in powers of g and ε while they are diverging in Taylor series of z and Laurent series of ε . For example, the scaling functions⁽⁵⁾

$$\sigma_0(z, \varepsilon) = \sigma_0[g, \varepsilon] = S \frac{\partial}{\partial S} \ln \mathcal{X}_0(z, d) = \frac{\varepsilon}{2} z \frac{\partial}{\partial z} \ln \mathcal{X}_0(z, d)$$

$$2\sigma_1(z, \varepsilon) = 2\sigma_1[g, \varepsilon] = S \frac{\partial}{\partial S} \ln \mathcal{Z}(\mathcal{S}_1) = \frac{\varepsilon}{2} z \frac{\partial}{\partial z} \ln Z(\mathcal{S}_1, z, d)$$

or the Wilson function⁽⁵⁾

$$W(z, \varepsilon) = W[g, \varepsilon] = S \frac{\partial}{\partial S} g = \frac{\varepsilon}{2} z \frac{\partial}{\partial z} g$$

become double series of g and ε , regular when $\varepsilon \rightarrow 0$, once expressed in terms of g [Eq. (3.10)] substituted for z . This direct polymer renormalization formalism can be proven^(34,44) to hold true to all orders in ε from the renormalization of the field theory $(\varphi^2)^2$. The second virial coef-

ficient g reaches the fixed point limit⁽⁵⁾ for $z \rightarrow \infty$, obtained from $W[g^*, \varepsilon] = 0$:

$$g^*(\varepsilon) = \frac{\varepsilon}{8} + \left(\frac{\varepsilon}{8}\right)^2 \left(\frac{25}{4} + 4 \ln 2\right) + \dots \quad (3.11)$$

I proposed a simpler minimal renormalization scheme for polymers in refs. 35 and 43. There it is sufficient to substitute for z in any scaling function a minimally renormalized parameter z_R of the form

$$z_R = z + \sum_{n \geq 2} z^n \left(\frac{a_n}{\varepsilon^n} + \dots + \frac{a_1}{\varepsilon}\right) \quad (3.12)$$

such that any scaling function becomes pole-free in ε , to all orders in z_R . It reads to first order, (for polymers)^(35,43)

$$z_R = z - z^2 \frac{8}{\varepsilon} + z^3 \left(\frac{64}{\varepsilon^2} + \frac{17}{\varepsilon}\right) + \dots \quad (3.12\text{bis})$$

and its fixed point value is

$$z_R^* = \frac{\varepsilon}{8} + \left(\frac{\varepsilon}{8}\right)^2 \frac{17}{4} + \dots \quad (3.12\text{ter})$$

I used this to calculate critical amplitudes in the Edwards model⁽⁴²⁾ or exact properties in $d=4$ dimensions⁽⁴³⁾ and shall rely on it later. Let us generalize the direct renormalization theory to any network.

3.2. Renormalization Principle for Networks: Factorization over the Vertices

First, note that the partition function $\mathcal{Z}(\mathcal{G})$ [Eq. (2.14)] of a given network \mathcal{G} scales naively as $S^{-\mathcal{L}d/2}$. But in the asymptotic limit, i.e., near the fixed point limit, the physical size will be given by R^2 [(3.3), (3.5)] and not by S anymore. So it is natural to set

$$\mathcal{Z}(\mathcal{G}, b, S, d) = (2\pi R^2/d)^{-d\mathcal{L}/2} Z_v(\mathcal{G}, z, d) \quad (3.13)$$

where Z_v is simply related to Z [defined in (2.14)] by

$$Z_v(\mathcal{G}, z, d) = [\mathcal{X}_0(z, d)]^{d\mathcal{L}/2} Z(\mathcal{G}, z, d)$$

Now, we have factorized out the hyperscaling contribution of the internal loops of the polymer network. Note, indeed, that for a single polymer loop

$\mathcal{L} = 1$, we factorize out $R^{-d} \sim S^{-vd} \sim S^{\alpha-2}$, where α is the usual specific heat exponent of the equivalent magnetic system. So what we are really doing here is stating a *generalized hyperscaling*. This is not sufficient. The polymer network still has vertices where new anomalous dimensions appear, as in field theory new anomalous dimensions appear for each power $\varphi^L(x)$ of, e.g., the $(\varphi^2)^2$ field theory. So we want to factorize out the contribution of the cores of the vertices. For this, we consider⁽¹⁾ the partition functions $\mathcal{Z}(\mathcal{S}_L)$ (in dimensional regularization) of the monodisperse star polymers \mathcal{S}_L of L -legs. They read explicitly, since there are no constitutive loops in a star, according to the general equation (2.14),

$$\mathcal{Z}(\mathcal{S}_L, b, S, d) \equiv Z(\mathcal{S}_L, z, d) \tag{3.14}$$

The idea⁽¹⁾ is now that in $\mathcal{Z}(\mathcal{S}_L)$, and more generally in the reduced vertex partition functions $Z_v(\mathcal{G})$, the divergences come from the vertices and factorize out over the latter. So we introduce a reduced partition function associated uniquely with the core of the L -star⁽¹⁾

$$\begin{aligned} \hat{Z}_L(z, d) &\equiv \mathcal{Z}(\mathcal{S}_L) [\mathcal{Z}(\mathcal{S}_1)]^{-L/2} \\ &= Z(\mathcal{S}_L, z, d) [Z(\mathcal{S}_1, z, d)]^{-L/2} \end{aligned} \tag{3.15}$$

This core partition function or renormalization factor depends only on z and d . The division by $[\mathcal{Z}(\mathcal{S}_1)]^{L/2}$ corresponds to the defactorization of the divergences of L free extremities of the L -star. Indeed, the 1-star is simply a linear chain with two extremities and Eq. (3.15) gives

$$\hat{Z}_1(z, d) \equiv [\mathcal{Z}(\mathcal{S}_1)]^{1/2} = Z^{1/2}(\mathcal{S}_1, z, d) \tag{3.16}$$

Note that a similar idea appears in ref. 5, where the renormalization factor \mathcal{X}_1 of a single-chain extremity is just \hat{Z}_1 here.

Now, we consider a general network \mathcal{G} , made of n_L vertices of L legs, $L \geq 1$. We can always rewrite its partition function under the form⁽¹⁾

$$\mathcal{Z}(\mathcal{G}, b, S, d) = [2\pi S \mathcal{X}_0(z, d)]^{-d\mathcal{L}/2} \prod_{L \geq 1} [\hat{Z}_L(z, d)]^{n_L} \mathcal{A}(\mathcal{G}, z, d) \tag{3.17}$$

where the amplitude $\mathcal{A}(\mathcal{G}, z, d)$ is a dimensionless function of z and d . This means that we have factorized out first the hyperscaling loop factor and then all the irreducible contributions (3.15) of the cores of L stars present in the graph \mathcal{G} . More explicitly, in terms of the partition functions of the stars of \mathcal{G} we have

$$\mathcal{Z}(\mathcal{G}) = (2\pi R^2/d)^{-d\mathcal{L}/2} \prod_{L \geq 1} [\mathcal{Z}(\mathcal{S}_L) \mathcal{Z}^{-L/2}(\mathcal{S}_1)]^{n_L} \mathcal{A}(\mathcal{G}) \tag{3.18}$$

Until now, all of this has been only a matter of definitions. The main step is now to state that all divergences in $1/\varepsilon$, or equivalently any anomalous scaling behavior in powers of S , have been factorized out. Hence, *the residual amplitude $\mathcal{A}(\mathcal{G}, z, d)$ reaches a finite fixed point value in the asymptotic limit $S \rightarrow \infty$, or $z \rightarrow \infty$:*

$$\mathcal{A}(\mathcal{G}, z, d) \xrightarrow{z \rightarrow \infty} \mathcal{A}^*(\mathcal{G}, \infty, d) < \infty \tag{3.19}$$

Furthermore, \mathcal{A}^* is *universal*. In terms of the ε -expansion, this means that $\mathcal{A}(\mathcal{G}, z, d)$, once reexpressed in terms of the *renormalized* dimensionless interactions g [(3.10)]⁽⁵⁾ or z_R [(3.12)]⁽³⁵⁾ instead of z , has a double Taylor series expansion in powers of g or z_R , and ε , which is regular for $\varepsilon \rightarrow 0$, to all orders in g or z_R :

$$\begin{array}{ccccc} \mathcal{A}(\mathcal{G}, z, d) & \equiv & \mathcal{A}[\mathcal{G}, g, d] & \equiv & \mathcal{A}[\mathcal{G}, z_R, d] \\ \text{singular in } 1/\varepsilon & & \text{regular} & & \text{regular} \end{array}$$

and its fixed point limit is

$$\mathcal{A}^* = \mathcal{A}(z \rightarrow \infty) = \mathcal{A}[g^*] = \mathcal{A}[[z_R^*]] \tag{3.20}$$

The statement (3.17), (3.19) is a nontrivial one.⁽¹⁾ It means that in addition to the hyperscaling factorization (3.13), all divergences of the field theory associated with a given network \mathcal{G} come from the vertices of \mathcal{G} and can be multiplicatively factorized over the vertices. Let us check it for the watermelon network.

3.3. Example of the Watermelon Network

The watermelon network \mathcal{W}_L has $L - 1$ constitutive loops (Fig. 5). We calculated its partition function to first order in (2.34), adding one loop via one interaction:

$$\begin{aligned} \mathcal{Z}(\mathcal{W}_L) &= (2\pi S)^{-d/2(L-1)} L^{-d/2} \\ &\times \left\{ 1 - z \left[\frac{2}{\varepsilon} (L^2 - 2) + 2 - 4L + L^2 + (L^2 - 2) \ln L \right] \right\} \end{aligned} \tag{3.21}$$

To renormalize it as in (3.18), we need the swelling factor (3.4) to first order

$$\mathcal{X}_0(z, \varepsilon) = 1 + z \left(\frac{2}{\varepsilon} - 1 \right) + \dots \tag{3.22}$$

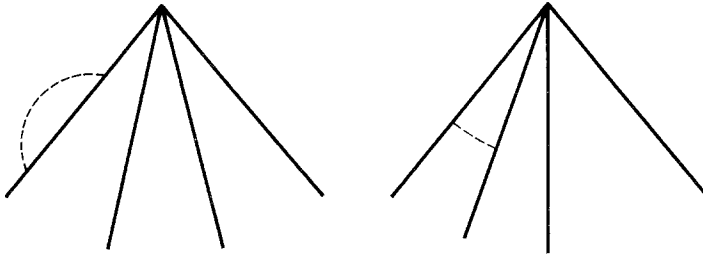


Fig. 7. First-order diagrams contributing to the star partition function $\mathcal{Z}(\mathcal{S}_L)$.

and the partition function $\mathcal{Z}(\mathcal{S}_L)$ of the L -star. It has been calculated in ref. 1 (see also ref. 5) to first order (Fig. 7) and an immediate application of the perturbative rules of Section 2.4 gives

$$\mathcal{Z}(\mathcal{S}_L) = 1 + L\mathcal{J} + \frac{1}{2}L(L-1)\mathcal{I} \tag{3.23}$$

where \mathcal{J} and \mathcal{I} correspond to the two diagrams of Fig. 7:

$$\begin{aligned} \mathcal{J} &= -b(2\pi)^{-d/2} \int_0^S ds (S-s) s^{-d/2} \\ &= -z \frac{\Gamma(1-d/2)}{\Gamma(3-d/2)} = -z \frac{1}{(2-d/2)(1-d/2)} \end{aligned}$$

Expanded in ε , it reads

$$\mathcal{J} = z[2/\varepsilon + 1 + O(\varepsilon)] \tag{3.24}$$

On the other hand,

$$\begin{aligned} \mathcal{I} &= -b(2\pi)^{-d/2} \int_0^S ds \int_0^S ds' (s+s')^{-d/2} \\ &= -z \frac{1}{(1-d/2)(2-d/2)} (2^{2-d/2} - 2) \end{aligned}$$

and in ε

$$\mathcal{I} = z(-2/\varepsilon + \ln 2 - 1) \tag{3.25}$$

Hence, to this order, the vertex renormalization factor (3.15) reads

$$\begin{aligned} \hat{Z}_L(z, d) &= 1 + \frac{L}{2}\mathcal{J} + \frac{1}{2}L(L-1)\mathcal{I} \\ &= 1 + z \frac{L}{2} \left[\frac{2}{\varepsilon} (2-L) + 1 + (L-1)(\ln 2 - 1) \right] \end{aligned} \tag{3.26}$$

For a watermelon network \mathcal{W}_L , the set of numbers of L' -leg vertices is trivially $n_L = 2$, $n_{L' \neq L} = 0$ and the number of loops is $\mathcal{L} = L - 1$. Hence the (*universal*) amplitude \mathcal{A} defined in (3.17), (3.18) reads

$$\begin{aligned} \mathcal{A}(\mathcal{W}_L) &= \mathcal{Z}(\mathcal{W}_L) [\mathcal{Z}(\mathcal{S}_L)]^{-2} [\mathcal{Z}(\mathcal{S}_1)]^L (2\pi S\mathcal{X}_0)^{d(L-1)/2} \\ &= \mathcal{Z}(\mathcal{W}_L) \hat{Z}_L^{-2} (2\pi S\mathcal{X}_0)^{d(L-1)/2} \end{aligned} \quad (3.27)$$

Substituting to first order in z all the Taylor–Laurent expansions (3.21)–(3.26), we find

$$\mathcal{A}(\mathcal{W}_L, z, d) = L^{-d/2} \{1 - z[(L-1)(1 + L \ln 2) + (L^2 - 2) \ln L]\}$$

As expected, the poles in $1/\varepsilon$ have all identically *disappeared* [to order $O(z)$] from this amplitude, and this for any L . Moreover, this should work to all orders, provided z is replaced by the renormalized g [Eq. (3.10)] or z_R [Eq. (3.12)]. Hence the amplitude \mathcal{A} reads

$$\mathcal{A}[\mathcal{W}_L, g, d] = L^{-d/2} \{1 - g[(L-1)(1 + L \ln 2) + (L^2 - 2) \ln L] + O(g^2)\}$$

and its *universal* fixed point value is, according to (3.11),

$$\mathcal{A}^*(\mathcal{W}_L) = L^{-d/2} \left\{1 - \frac{1}{8}\varepsilon[(L-1)(1 + L \ln 2) + (L^2 - 2) \ln L] + O(\varepsilon^2)\right\} \quad (3.28)$$

This nontrivial example (to first order) illustrates our general scaling factorization (3.17) and (3.18) over vertices. We obtain indeed that a watermelon network, which is topologically quite distinct from stars, can be renormalized simply in terms of its two L -star vertices and in terms of the hyperscaling loop factor (3.13).

Admittedly, this statement remains to be established to all orders, and in field theory is *far* from being a trivial statement. However, I believe that in polymer theory, it is almost “obvious” that only the vertices of a network will play a role, provided the general overall dimensional factor has been properly taken into account by hyperscaling. Moreover, I give below another derivation^(13,22) of the fundamental equation (3.18) in *two dimensions*, where I use *conformal scaling*. As we shall see, Eq. (3.18) finds a natural expression in terms of the basic scaling dimensions x_L of the correlation functions (2.40) of Section 2.6.

The assumption of renormalization to all orders of Eq. (3.18) will appear in two dimensions as a simple equivalent assumption^(1,12,13,22) that the L -star polymer vertices correspond in the $O(n)$ conformal field theory to some primary vertex operators ϕ_L of conformal invariance theory⁽²⁷⁾ of dimensions x_L . Let us now fully exploit (3.18).

3.4. Basic Topological Exponents

Let us consider first the star polymer with L arms. Due to (1.4) and (2.5) we expect its dimensionless partition function (3.14) to scale in the asymptotic regime like

$$\mathcal{Z}(\mathcal{S}_L) \underset{z \rightarrow \infty}{=} Z(\mathcal{S}_L, z, d) = A_L(\varepsilon) z^{(\gamma_L - 1)2/\varepsilon} \sim S^{\gamma_L - 1} \quad (3.29)$$

where $A_L(\varepsilon)$ is a (calculable) amplitude. Accordingly, the proper irreducible partition function (3.15) \hat{Z}_L associated with the L -vertex scales for S or z large like

$$\hat{Z}_L(z, \varepsilon) \underset{z \rightarrow \infty}{\rightarrow} \hat{A}_L(\varepsilon) z^{\sigma_L 2/\varepsilon} \sim S^{\sigma_L} \quad (3.30)$$

where σ_L is a new universal exponent

$$\sigma_L = \gamma_L - 1 - \frac{1}{2}L(\gamma - 1) \quad (3.31)$$

and $\hat{A}_L = A_L A_1^{-L/2}$. The critical exponents γ_L or σ_L are universal and independent. Each of these corresponds to a new scaling operator depending on $L = 1, 2, \dots$; γ_L is really the total configuration exponent of the L arm-star including the L extremities, while σ_L is *more fundamental* and corresponds exactly to the anomalous scaling behavior of the vertex.

Now, from the general equation (3.17) one can read off the critical exponent $\gamma_{\mathcal{G}}$ of (2.6) associated with any polymer network. Using (3.5), (3.19), (3.30), we find identically the exact asymptotic behavior

$$\mathcal{Z}(\mathcal{G}) \underset{\substack{z \rightarrow \infty \\ S \rightarrow \infty}}{=} [2\pi S A_0(\varepsilon) z^{(2\nu - 1)2/\varepsilon}]^{-d\mathcal{L}/2} \prod_{L \geq 1} [\hat{A}_L(\varepsilon) z^{\sigma_L 2/\varepsilon}]^{n_L} \mathcal{A}^*(\mathcal{G}) \quad (3.32)$$

In terms of S , we find simply⁽¹⁾

$$\mathcal{Z}(\mathcal{G}) \sim S^{-\nu d\mathcal{L} + \sum_{L \geq 1} n_L \sigma_L}, \quad S \rightarrow \infty \quad (3.33)$$

Hence, as announced,⁽¹⁾

$$\gamma_{\mathcal{G}} - 1 = -\nu d\mathcal{L} + \sum_{L \geq 1} n_L \sigma_L \quad (3.34)$$

I stress that this (hyper)scaling relation works in any dimension,⁽¹⁾ in particular in 2D or in $d = 4 - \varepsilon$. It is sufficient to determine the basic set of topological critical exponents σ_L for knowing any $\gamma_{\mathcal{G}}$ for any network. *So the polymer network problem has been entirely reduced to considering star polymers.* The latter are far simpler objects than general networks, and the

calculation of their critical exponents γ_L or σ_L is simple.^(1,5) Note also that due to Eq. (2.13), we can write also trivially

$$\begin{aligned}\gamma_{\mathcal{G}} - 1 &= -vd + \sum_{L \geq 1} n_L \Delta_L \\ \Delta_L &\equiv \sigma_L - \frac{1}{2}vd(L-2)\end{aligned}\tag{3.34bis}$$

Note also that the usual exponent γ of a single chain is simply $\gamma_{L=1}$ and from (3.31) or (3.34)

$$\gamma - 1 = 2\sigma_1\tag{3.35}$$

Consequently, the exponent γ_L of a simple L -arm star polymer is, from (3.31),

$$\gamma_L - 1 = \sigma_L + L\sigma_1\tag{3.36}$$

Of course, using (3.31), one can rewrite any exponent $\gamma_{\mathcal{G}}$ in terms of the γ_L exponents of the stars present in \mathcal{G} , by substituting γ_L in place of σ_L . Hence Δ_L in (3.34bis) can be written

$$\Delta_L = \gamma_L - 1 - \frac{1}{2}L(\gamma - 1) - \frac{1}{2}vd(L-2)\tag{3.34ter}$$

The σ_L are known exactly in two dimensions [Eq. (1.15)] from conformal invariance theory and Coulomb-gas techniques.^(1,12,16,17) They are also known to $O(\varepsilon)$ in $d=4-\varepsilon$,^(1,5) and we calculate their $O(\varepsilon^2)$ value in the next sections.

The basic method for obtaining the σ_L in $d=4-\varepsilon$ is first to calculate the perturbation expansion of the star partition functions $\mathcal{Z}(\mathcal{S}_L)$, thus \hat{Z}_L of (3.15), and to introduce the scaling functions

$$\sigma_L(z, \varepsilon) = S \frac{\partial}{\partial S} \ln \hat{Z}_L(z, d) = \frac{\varepsilon}{2} z \frac{\partial}{\partial z} \ln \hat{Z}_L(z, d)\tag{3.37}$$

Once substituted in terms of g or z_R

$$\sigma_L(z, \varepsilon) = \sigma_L[g, \varepsilon] = \sigma_L[[z_R, \varepsilon]]\tag{3.38}$$

they become regular double series expansions of g, ε or z_R, ε . Their fixed point values as usual yield the critical exponents σ_L ,

$$\sigma_L \equiv \sigma_L(z \rightarrow \infty, \varepsilon) = \sigma_L[g^*, \varepsilon] = \sigma_L[[z_R^*, \varepsilon]]\tag{3.39}$$

Using (3.26), we find easily to first order

$$\begin{aligned}\sigma_L(z, \varepsilon) &= z \frac{L}{2} (2-L) + O\left(\frac{z^2}{\varepsilon}\right) \\ &= \sigma_L[g, \varepsilon] = g \frac{L}{2} (2-L) + O(g^2)\end{aligned}\tag{3.40}$$

with a fixed point value

$$\sigma_L = \frac{\varepsilon L}{8} (2 - L) + O(\varepsilon^2) \tag{3.41}$$

Note also that the scaling behavior above four dimensions is trivial. Indeed, we have identically

$$\sigma_L = 0 \quad \text{for } d > 4 \tag{3.42}$$

and $\nu = 1/2$. Hence the network partition function (3.33), (3.34) scales like

$$\mathcal{Z}(\mathcal{G}) \sim (2\pi S)^{-d\nu/2}$$

which is just the Brownian behavior (2.15), as expected.

It is also interesting to look at the marginal case $d = 4$.^(28,43)

3.5. Logarithmic Behavior in $d = 4$

The factorization formula (3.17) still holds as in any d . The only difference is that the factors have various scaling behaviors. First we have in 4D⁽⁴³⁾

$$R^2 \sim S(\ln S)^{1/4} \tag{3.43}$$

Second, the scaling function (3.37) reads

$$S \frac{\partial}{\partial S} \ln \hat{Z}_L = \sigma_L[g, 0] = g \frac{L}{2} (2 - L) \tag{3.44}$$

where g in 4D has a logarithmically vanishing value⁽⁴³⁾

$$g = \frac{1}{4 \ln S} + \frac{17 \ln(4 \ln S)}{4 (4 \ln S)^2} + O\left(\frac{1}{\ln^2 S}\right) \tag{3.45}$$

and, in principle, a cutoff term s_0 is present⁽⁴³⁾ in $\ln(S/s_0)$, not recalled here, for simplicity. Substituting (3.45) in the equation above and integrating on S gives

$$\hat{Z}_L \sim (\ln S)^{L(2-L)/8} \left[1 + O\left(\frac{\ln \ln S}{\ln S}\right) \right] \tag{3.46}$$

up to calculable corrections.^(28,43)

Substituting the logarithmic terms (3.43) and (3.46) into (3.17) gives for any network the exact behavior

$$\mathcal{Z}(\mathcal{G}) \underset{\substack{d=4 \\ S \rightarrow \infty}}{\sim} (2\pi S)^{-2\mathcal{L}} (\ln S)^{-\mathcal{L}/2 + \sum_{L \geq 1} n_L L(2-L)/8} \tag{3.47}$$

Here again, corrections terms of order $\ln \ln S / \ln S$ are easily calculable (see Section 6.c of ref. 28). Note also that the number of loops can be expressed in terms of the $\{n_L\}$ [Eq. (2.13)]. Hence, in 4D, the exact logarithmic scaling for a polymer network with excluded volume reads finally

$$\mathcal{Z}(\mathcal{G}) \sim (2\pi S)^{-2\mathcal{L}} (\ln S)^{-1/2 + \sum_{L \geq 1} n_L(4-L^2)/8} \tag{3.48}$$

For a star polymer with L arms, $n_1 = L$, $n_L = 1$, $n_{L' \neq L} = 0$, and $\mathcal{L} = 0$. Hence (3.47) gives trivially the exact dominant behavior

$$\mathcal{Z}(\mathcal{S}_L) \underset{d=4}{\sim} (\ln S)^{-L(L-3)/8} \tag{3.47bis}$$

3.6. Link to the Anomalous Dimensions x_L

The critical exponents σ_L correspond to the scaling of the L -leg vertices in the “size space” S , while in Section 2.6 we defined some critical anomalous dimensions x_L [Eq. (2.40)] associated with the algebraic decay in space of the field-theoretic correlation function of L polymer lines tied together, and with fluctuating lengths (Fig. 5). Of course, the sets $\{\sigma_L\}$ and $\{x_L\}$ are related one to another.^(1,18)

For this we first calculate the polydisperse partition function $\mathcal{Z}_L^{\text{poly}}(S)$, (2.38), of the “watermelon,” using in the integral the scaling form (2.43) of G_L away from criticality. We find after integration of (2.38) and a simple dimensional analysis

$$\mathcal{Z}_L^{\text{poly}}(S) \sim e^{T_c S} S^{-v(2x_L - d) - 1} \tag{3.49}$$

which gives a *polydisperse* exponent $\gamma_{\mathcal{W}_L}^{\text{poly}}$

$$\gamma_{\mathcal{W}_L}^{\text{poly}} = -v(2x_L - d) \tag{3.50}$$

Note that $e^{T_c} = \mu$ is just the effective connectivity constant, regularization dependent.

Now the relation (2.39) holds⁽¹⁾ between the polydisperse watermelon partition function with lengths fluctuating about a total length S and the monodisperse one where all the L lengths equal S . Hence the monodisperse exponent $\gamma_{\mathcal{W}_L}$ of the watermelon network is simply, from (3.50),

$$\gamma_{\mathcal{W}_L} - 1 = -v(2x_L - d) - L \tag{3.51}$$

Now, we can apply⁽¹⁾ the general formula (3.34) to this particular network \mathscr{W}_L described by the simple set $\{n_L\}$

$$n_L = 2, \quad n_{L' \neq L} = 0 \tag{3.52}$$

which is a *pure L-type case*, with two vertices. Hence

$$\gamma_{\mathscr{W}_L} - 1 = -vd(L - 1) + 2\sigma_L$$

Comparing this relation to (3.51) yields

$$\sigma_L = -vx_L + L(vd - 1)/2 \tag{3.53}$$

or, in terms of the anomalous part x'_L , (2.40b),

$$\sigma_L = -vx'_L + L(v - 1/2) \tag{3.53bis}$$

This formula is important since it relates two really distinct topologies, namely that of pure *L*-stars (σ_L) and that of watermelons (x_L) where the extremities of the *L* arms are tied. It is also technically interesting since either σ_L or x_L is the easiest accessible quantity. In $4 - \varepsilon$ dimensions, the σ_L are the best quantities to evaluate (see Section 4), while in two dimensions, the dimensions x_L of the *L*-vertex operator belong to the Kac table⁽⁴⁵⁾ of conformal invariance, and are accessible from Coulomb-gas methods.⁽¹⁶⁻²¹⁾ Finally, we can rewrite the general formula (3.34bis) in terms of the x_L ,

$$\begin{aligned} \gamma_{\mathscr{G}} - 1 &= -vd + \sum_{L \geq 1} n_L \Delta_L \\ \Delta_L &= v(d - x_L) - L/2 \\ &= v(d - x'_L) - L/2 - v(d - 2)L/2 \end{aligned} \tag{3.54}$$

Note finally an important consistency check⁽¹⁾ of the scaling theory presented here. As already mentioned, the insertion of 2-leg vertices on the network is just a trivial point insertion, which divides a chain into two connected pieces, and hence does not change the network \mathscr{G} . Hence, n_2 is really arbitrary in \mathscr{G} , and can even be interpreted as the number of monomers of \mathscr{G} ! Hence, we must have identically

$$\sigma_2 = 0 \tag{3.55}$$

This comes also mathematically from the identity $\gamma_2 = \gamma_1 \equiv \gamma$ of the γ exponents of a two-leg star and of a single chain. Inserting this in (3.31) gives indeed $\sigma_2 = 0$. This identity is checked in the exact two-dimensional

value (1.15) and to order $O(\varepsilon)$ in (3.41). Notice then that $\sigma_2 = 0$ implies in (3.53) the identity

$$d - x_2 = 2 - x'_2 = 1/\nu \quad (3.56)$$

This result is actually just another form of the hyperscaling relation $\alpha = 2 - \nu d$, and can be obtained directly from the definition of the correlator G_2 [(2.36), (2.40)], which in field theory, or in the $O(n)$ model, is the energy–energy correlator.

4. CONFORMAL INVARIANCE IN TWO DIMENSIONS

Let us give an alternative derivation^(22,13) of the renormalization equation (3.17) for two-dimensional polymer networks. In conformal invariance theory we associate^(12,13,22) a (primary) operator^(27,46) ϕ_L to each L -leg vertex in a polymer [or, more generally, to each L -leg vertex in the $O(n)$ model; see refs. 20 and 26 for a discussion in this case]. Then the correlator G_L [(2.36), (2.40)] of the two L -leg vertices in the watermelon network reads

$$\langle \phi_L(\mathbf{X}) \phi_L(\mathbf{Y}) \rangle_T = G_L(\mathbf{X} - \mathbf{Y}, T) \quad (4.1)$$

and according to Eq. (2.40) it decays algebraically at the critical point for $d = 2$ as

$$\langle \phi_L(\mathbf{X}) \phi_L(\mathbf{Y}) \rangle_{T_c} = |\mathbf{X} - \mathbf{Y}|^{-2x_L} \quad (4.2)$$

where x_L appears as the scaling dimension of the vertex operator ϕ_L .

Now consider a network \mathcal{G} made of vertices $i = 1, \dots, \mathcal{V}$, with L_i legs, and placed at position z_i in the complex plane, and described by the vertex operators $\phi_{L_i}(z_i)$. Then the field-theoretic correlation function of these ϕ_L is related by Laplace transform to the multiple correlation function of the network⁽²²⁾

$$G_{\mathcal{G}}(z_1, \dots, z_{\mathcal{V}}; T) \equiv \left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_T \quad (4.3)$$

$$= \int_0^\infty \prod_{a=1}^{\mathcal{N}} dS_a \exp \left[-T \sum_{a=1}^{\mathcal{N}} S_a \right] \\ \times \mathcal{Z}_{\mathcal{G}}(S_1, \dots, S_{\mathcal{N}}; z_1, \dots, z_{\mathcal{V}}) \quad (4.4)$$

Note that here the total length is fluctuating with a chemical potential $-T$, and that the positions of the vertices are fixed in the plane. The network \mathcal{G} is first taken as polydisperse with lengths $S_1, \dots, S_{\mathcal{N}}$ in the \mathcal{N} branches.

At the critical point $e^{T_c} = \mu^{-1}$, the critical correlation function has the properties derived from conformal invariance theory.^(27,46) $G_{\mathcal{G}}$ is *conformally covariant*. If $w(z)$ is any analytic transformation of the complex plane, we have

$$\left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_{T_c} = \prod_{i=1}^{\mathcal{V}} |w'(z_i)|^{x_{L_i}} \left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(w(z_i)) \right\rangle_{T_c} \tag{4.5}$$

where $w'(z) = dw(z)/dz$ and where the x_{L_i} are the scaling dimensions (4.2) of the vertex operators ϕ_{L_i} . Applying this quite general conformal covariance equation, one can show that the multipoint correlation functions have the form^(27,46) (for scalar primary operators)

$$\left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_{T_c} = \prod_{\substack{i,j=1 \\ i \neq j}}^{\mathcal{V}} |z_i - z_j|^{-\Delta_{ij}} \mathcal{F} \left\{ \frac{|z_i - z_j|}{|z_k - z_l|}, i, j, k, l = 1, \dots, \mathcal{V} \right\} \tag{4.6}$$

The values of Δ_{ij} can be computed in terms of the scaling dimensions x_{L_i} of the operators ϕ_{L_i} and are linear combinations of the latter. For instance, in the case of a three-point function one has simply^(27,46)

$$\begin{aligned} &\langle \phi_{L_1}(z_1) \phi_{L_2}(z_2) \phi_{L_3}(z_3) \rangle \\ &= |z_1 - z_2|^{-x_1 - x_2 + x_3} |z_2 - z_3|^{-x_2 - x_3 + x_1} |z_3 - z_1|^{-x_3 - x_1 + x_2} \end{aligned}$$

and $\mathcal{F} \equiv 1$. Hence $\Delta_{12} = x_1 + x_2 - x_3$ and similar permutations for Δ_{23} , Δ_{31} . Thus, the total scaling dimension is $\Delta_{12} + \Delta_{23} + \Delta_{31} = x_1 + x_2 + x_3$. We shall need the *overall* scaling dimension of $\langle \prod_i \phi_{L_i} \rangle$ for a global dilation. This dimension is, from (4.6), the sum $\mathcal{S} = \sum_{i,j=1, i \neq j}^{\mathcal{V}} \Delta_{ij}$. It can be shown,⁽⁴⁶⁾ as above, to be simply the sum of the individual scaling dimensions of the operators ϕ_{L_i} . Hence

$$\mathcal{S} = \sum_{i=1}^{\mathcal{V}} x_{L_i} \tag{4.7}$$

Now, we consider the correlation function (4.4), (4.6) slightly off criticality, for $T \neq T_c$. Then we expect it to scale like

$$\begin{aligned} \left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_T &= \prod_{\substack{i,j=1 \\ i \neq j}}^{\mathcal{V}} |z_i - z_j|^{-\Delta_{ij}} \\ &\times \mathcal{F} \left\{ \frac{|z_i - z_j|}{|z_k - z_l|}, |z_i - z_j| |T - T_c|^{\nu}, i, j, k, l = 1, \dots, \mathcal{V} \right\} \end{aligned} \tag{4.8}$$

Now, the *polydisperse* correlation function of the network \mathcal{G} with fixed total length S can be obtained by inverting (4.4) and using (4.8),

$$\begin{aligned} \mathcal{Z}_{\mathcal{G}}^{\text{poly}}(S; z_1, \dots, z_{\mathcal{V}}) \\ = \int_0^\infty \prod_{a=1}^{\mathcal{N}} dS_a \delta\left(S - \sum_{a=1}^{\mathcal{N}} S_a\right) \mathcal{Z}_{\mathcal{G}}(S_1, \dots, S_{\mathcal{N}}; z_1, \dots, z_{\mathcal{V}}) \end{aligned} \quad (4.9)$$

and reads

$$\mathcal{Z}_{\mathcal{G}}^{\text{poly}}(S; z_1, \dots, z_{\mathcal{V}}) = \int \frac{dT}{2\pi i} e^{TS} \left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_T \quad (4.10)$$

where the contour is taken in the complex plane from $-i\infty$ to $+i\infty$, at the right of the T_c singularity. The *total* polydisperse partition function is obtained by integrating the z_i in (4.10) over the complex plane, except for one, which is fixed for eliminating translational invariance,

$$\mathcal{Z}_{\mathcal{G}}^{\text{poly}}(S) = \int \frac{dT}{2\pi i} e^{TS} \int \prod_{i=1}^{\mathcal{V}-1} d^2 z_i \left\langle \prod_{i=1}^{\mathcal{V}} \phi_{L_i}(z_i) \right\rangle_T \quad (4.11)$$

One inserts now the general form (4.8) in (4.11) and performs simple dimensional analysis for extracting the scaling power of (4.11) in terms of S . Using (4.7) gives readily

$$\mathcal{Z}_{\mathcal{G}}^{\text{poly}}(S) \sim e^{T_c S} S^{-1 + \nu[2(\mathcal{V}-1) - \sum_{i=1}^{\mathcal{V}} n_{L_i} x_{L_i}]} \quad (4.12)$$

and hence a polydisperse exponent

$$\gamma_{\mathcal{G}}^{\text{poly}} = \nu \left[2(\mathcal{V}-1) - \sum_{L \geq 1} n_L x_L \right] \quad (4.13)$$

This holds for the integrated partition function (4.9) of a network \mathcal{G} where only $S = \sum_{a=1}^{\mathcal{N}} S_a$ is fixed. If one goes to a monodisperse one where all $S_a = S$, $a = 1, \dots, \mathcal{N}$, one has simply to factorize out⁽¹⁾ the constraint factor $S^{\mathcal{N}-1}$ appearing in (4.9). Hence our usual partition function (2.2) reads

$$\mathcal{Z}(\mathcal{G}) \sim e^{T_c S} S^{\gamma_{\mathcal{G}} - 1}, \quad S \rightarrow \infty$$

with

$$\gamma_{\mathcal{G}} - 1 = \gamma_{\mathcal{G}}^{\text{poly}} - \mathcal{N} = \nu \left[2(\mathcal{V}-1) - \sum_{L \geq 1} n_L x_L \right] - \mathcal{N} \quad (4.14)$$

Of course, the derivation given here in two dimensions is quite general. In d dimensions one can see easily from (2.40) and (4.11) that the generalization of (4.14) is

$$\gamma_{\mathcal{G}} - 1 = v \left[d(\mathcal{V} - 1) - \sum_{L \geq 1} n_L x_L \right] - \mathcal{N} \tag{4.14bis}$$

where

$$x_L \equiv x'_L + \frac{L}{2}(d-2) \tag{4.14ter}$$

is the *total* scaling dimension of the L -vertex operator in (2.40). Now, we can use the trivial topological relations $\mathcal{N} = \sum_L n_L L/2$ and $\mathcal{V} = \sum_L n_L$ to get in (4.14) in *two dimensions*

$$\gamma_{\mathcal{G}} - 1 = -2v + \sum_{L \geq 1} n_L [v(2 - x_L) - L/2] \tag{4.15}$$

Similarly, (4.14bis) just gives in d dimensions

$$\begin{aligned} \gamma_{\mathcal{G}} - 1 &= -vd + \sum_{L \geq 1} n_L \{v[d - x'_L - (d-2)L/2] - L/2\} \\ &= -vd + \sum_{L \geq 1} n_L \{v(d - x_L) - L/2\} \end{aligned} \tag{4.15bis}$$

which is just (3.54), derived there from the renormalization by stars [Eq. (3.17)]. So we see that the description of a network in the Edwards model via the partition function (2.2) and the renormalization principles (3.18), on one hand, or the field-theoretic $O(n)$ description (4.3), augmented by the structure of multipoint correlation functions (4.8), on the other hand, are entirely equivalent, as they must be. This reinforces strongly our belief in the validity of the multiplicative renormalization scheme (3.17) to all orders. In two dimensions the values of scaling dimensions x_L belong to the so-called conformal Kac table.^(27,45,46) See refs. 12, 13, 18, and 20 for detailed studies. It has been shown⁽⁴⁶⁾ that unitary critical models with a central charge⁽²⁷⁾ $c < 1$ can be classified as

$$c = 1 - 6/m(m+1), \quad m \geq 3 \tag{4.16}$$

where m is an integer, and with scaling dimensions x of the scalar “primary” conformal operators

$$x = 2h_{p,q}$$

where $h_{p,q}$ belongs to the Kac table⁽⁴⁵⁾

$$h_{h,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad (4.17)$$

where p, q are integers in the minimal block⁽⁴⁶⁾

$$1 \leq p \leq m-1, \quad 1 \leq q \leq p \quad (4.18)$$

The central charge is a *universal* number associated with the finite-size corrections to the bulk critical free energy, and is characteristic of a given critical theory.⁽²⁷⁾ For polymers, the free energy of the $n=0$ model is always zero, and hence $c=0$.^(12,13) Thus, one finds in (4.16), $m=2$. Then the scaling dimensions x_L of the L -vertex operators (2.40) have been numerically identified as⁽¹²⁾

$$x_L = 2h_{L/2,0} = (9L^2 - 4)/48 \quad (4.19)$$

So one sees that polymers in 2D correspond to a $c=0$ conformal field theory, with $m=2$, i.e., this theory is just preceding the $m \geq 3$ classification of Friedan *et al.*⁽⁴⁶⁾ Note also that for polymers, one has to extend the values of p, q outside the minimal block (4.18), which is here reduced to the trivial identity $h_{1,1}=0$. Moreover, p, q can be half-integers. The conformal formula (4.18) can be also derived by Coulomb-gas techniques^(16,17,20) and extended to polymers in the dense phase,^(13,18-20) at the Θ -point,⁽²⁶⁾ and near a surface⁽²²⁾ (see Section 8).

5. TOPOLOGICAL EXPONENTS σ_L TO ORDER $O(\epsilon^2)$

5.1. Diagrammatic Expansion

In this section, we calculate the exponents σ_L , $L \geq 1$, to second order in $\epsilon = 4 - d$, and hence any $\gamma_\mathcal{G}$ by the same direct method as in ref. 28 for the critical exponents of the intersections of random walks. Note that in another context, namely that of diffusion near absorbing fractals,⁽⁴⁷⁾ some similar star exponents were calculated.

According to the general theory (3.15), (3.30) above, we have simply to evaluate the (dimensionless) partition function $\mathcal{Z}(\mathcal{S}_L)$ of an L -leg star polymer for any L , to *second order* in z , the first order being given in (3.23), (3.26). Then one will extract the exponents γ_L and σ_L by renormalizing the singular $(z, 1/\epsilon)$ expansion of $\mathcal{Z}(\mathcal{S}_L)$ or $\hat{\mathcal{Z}}_L$, as in (3.37)–(3.39).

The diagrams contributing to $\mathcal{Z}(\mathcal{S}_L)$ up to order z^2 are given in Fig. 8. Their contributions are calculated with the perturbative rules given in Section 2.4.

The two-loop diagrams of Fig. 8a involve interactions between different arms of the star, and already appeared in the theory of the intersections of Brownian paths,⁽²⁸⁾ i.e., of mutually avoiding walks, and I have taken them from my previous work. The second-order diagrams of Fig. 8b involve at least one self-interaction of a given arm, and are to be added to those of Fig. 8a in order to get the statistical mechanics of not only mutually avoiding but also self-avoiding polymer chains in a star. Con-

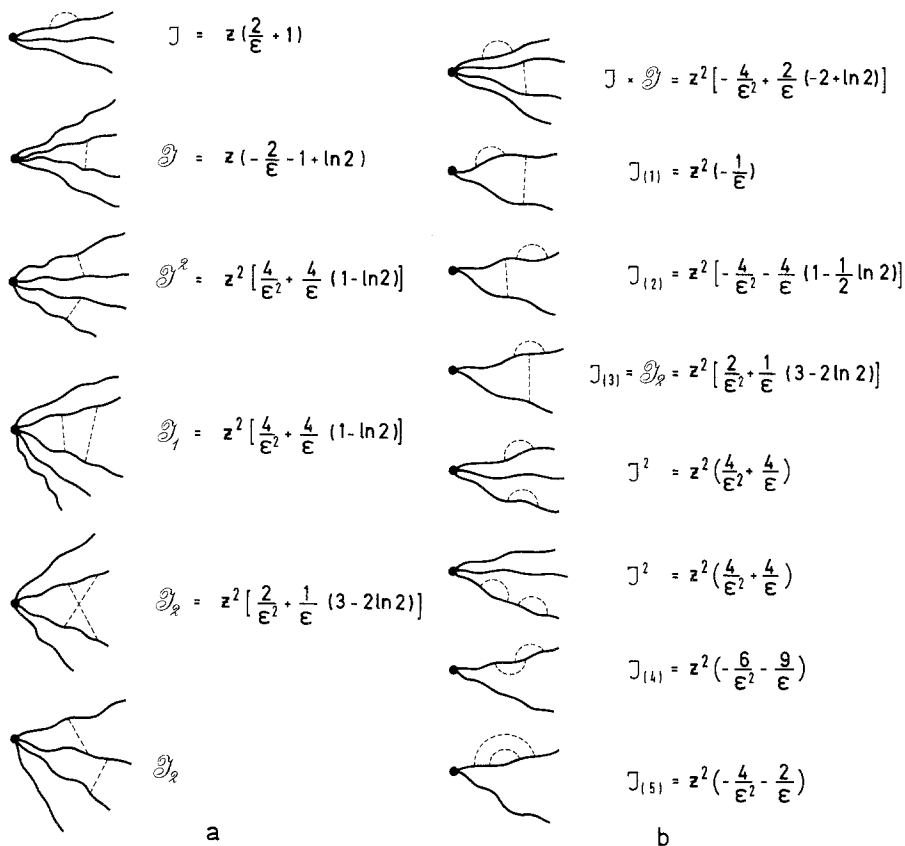


Fig. 8. Diagrams contributing up to two loops to the star partition function, and their elementary contributions. (a) Diagrams involving the mutual interactions of different arms (calculated in ref. 28 for the random walk intersection problem). (b) Diagrams involving self-interactions.

tributions $\mathcal{J}_{(1)}$ to $\mathcal{J}_{(3)}$ are new, while the four last diagrams of Fig. 8b of course appear in the theory of simple linear polymer chains.⁽⁵⁾

The evaluation of these diagrams requires some care, since they are polymer diagrams at fixed lengths S of all the arms. For instance, the integral representation of the diagram \mathcal{J}_2 of Fig. 8a is, according to the rules of Section 2.4,

$$\begin{aligned} \mathcal{J}_2 = z^2 \int_0^\infty ds_1 ds_2 ds'_1 ds'_2 \\ \times \theta(1-s_1-s_2) \theta(1-s'_1-s'_2) [(s_1+s'_1)(s_2+s'_2) + s_2 s'_2]^{-d/2} \end{aligned} \quad (5.1)$$

To extract its Laurent expansion in $1/\varepsilon$, the best way is to represent each θ function as an imaginary contour integral

$$\theta(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{da}{2\pi ia} e^{ax}, \quad \sigma \in \mathbb{R}^+ \quad (5.2)$$

This reconstructs the Schwinger–Feynman α -parameter representation of the associated field theory, where *two different* “masses” a occur, one for *each* arm. One has then to extend the standard treatments of such α integrals to the present more complicated case of multiple masses.² This procedure is also different from the one used in ref. 47, where the total length $S_1 + S_2$ of the two arms fluctuates, with a standard killing factor $\exp[-m^2(S_1 + S_2)]$.

I skip the technical details here and give the independent contributions.

1. One-loop:

$$\begin{aligned} \mathcal{J} &= z \left(\frac{2}{\varepsilon} + 1 \right) \\ \mathcal{J} &= z \left(-\frac{2}{\varepsilon} - 1 + \ln 2 \right) \end{aligned} \quad (5.3)$$

2. Two-loop:

$$\begin{aligned} \mathcal{J}_1 &= z^2 \left[\frac{4}{\varepsilon^2} + \frac{4}{\varepsilon} (1 - \ln 2) \right] \\ \mathcal{J}_2 &= z^2 \left[\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} (3 - 2 \ln 2) \right] \\ \mathcal{J}_{(1)} &= z^2 \left(-\frac{1}{\varepsilon} \right) \end{aligned}$$

² For a treatment in the symmetric one-mass case see, e.g., ref. 48.

$$\begin{aligned}
 \mathcal{J}_{(2)} &= z^2 \left[-\frac{4}{\varepsilon^2} - \frac{4}{\varepsilon} \left(1 - \frac{1}{2} \ln 2 \right) \right] \\
 \mathcal{J}_{(3)} &= z^2 \left[\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} (3 - 2 \ln 2) \right] \\
 \mathcal{J}_{(4)} &= z^2 \left(-\frac{6}{\varepsilon^2} - \frac{9}{\varepsilon} \right) \\
 \mathcal{J}_{(5)} &= z^2 \left(-\frac{4}{\varepsilon^2} - \frac{2}{\varepsilon} \right)
 \end{aligned} \tag{5.4}$$

Now, for a given number L of arms in the star \mathcal{S}_L , one has to weight each diagram according to the combinatorial number of times it appears when different arms interact. We find finally for any $L \geq 1$, in the order of Fig. 8,

$$\begin{aligned}
 \mathcal{Z}(\mathcal{S}_L) &= 1 + L\mathcal{J} + \binom{L}{2}\mathcal{J} + \frac{1}{2}\binom{L}{2}\binom{L-2}{2}\mathcal{J}^2 \\
 &+ \binom{L}{2}\mathcal{J}_1 + \binom{L}{2}\mathcal{J}_2 + 6\binom{L}{3}\mathcal{J}_2 \\
 &+ L\binom{L-1}{2}\mathcal{J}\mathcal{J} + L(L-1)\mathcal{J}_{(1)} + L(L-1)\mathcal{J}_{(2)} \\
 &+ L(L-1)\mathcal{J}_{(3)} + \binom{L}{2}\mathcal{J}^2 + L\mathcal{J}^2 + L\mathcal{J}_{(4)} + L\mathcal{J}_{(5)}
 \end{aligned} \tag{5.5}$$

Using now the values (5.3), (5.4) and regrouping terms, we find to this order

$$\mathcal{Z}(\mathcal{S}_L) = 1 + z \left(\frac{A_L}{\varepsilon} + A'_L \right) + z^2 \left(\frac{B_L}{\varepsilon^2} + \frac{B'_L}{\varepsilon} \right) \tag{5.6}$$

with

$$A_L = -2\binom{L}{2} + 2L = (3-L)L \tag{5.7}$$

$$A'_L = (\ln 2 - 1)\binom{L}{2} + L \tag{5.8}$$

$$B_L = 6\binom{L}{2} + 2\binom{L}{2}\binom{L-2}{2} - 6L \tag{5.9}$$

$$\begin{aligned}
 B'_L = & \binom{L}{2} (7 - 6 \ln 2) + 2L \binom{L-1}{2} (1 - \ln 2) \\
 & + 2(1 - \ln 2) \binom{L}{2} \binom{L-2}{2} - 7L
 \end{aligned} \tag{5.10}$$

Now, we have to renormalize this singular Laurent–Taylor expansion in order to extract the asymptotic behavior $\mathcal{Z}(\mathcal{S}_L) \sim S^{\gamma_L - 1}$.

5.2. Universal Values of the Exponents

Since we use the “direct renormalization” method,⁽⁵⁾ the calculations are very simple and require only the calculation of a log derivative like (3.37). (There are no explicit renormalization conditions as in field theory even if they are hidden in the process.^(34,35)) We could introduce $\sigma_L(z, \varepsilon)$, (3.37), associated with the reduced core partition function \hat{Z}_L , (3.15), which can be calculated to second order from (5.6). Here we choose first to work with the complete star partition function $\mathcal{Z}(\mathcal{S}_L)$. So we introduce the logarithmic derivative

$$\gamma_L(z, \varepsilon) - 1 = S \frac{\partial}{\partial S} \ln \mathcal{Z}(\mathcal{S}_L) = \frac{\varepsilon}{2} z \frac{\partial}{\partial z} \ln \mathcal{Z}(\mathcal{S}_L)|_\varepsilon \tag{5.11}$$

Its expression $\gamma_L[[z_R, \varepsilon]]$ once z_R [Eq. (3.12bis)] (or $g^{(5)}$) is substituted for z is a regular double Taylor series in z_R, ε (or g, ε). I shall not need here the straightforward but lengthy corresponding algebra for the particular values (5.7)–(5.10) associated with $\mathcal{Z}(\mathcal{S}_L)$, (5.6), since I have already given the general answer in ref. 42 for any scaling function of the form (5.6) (see Section 3 of ref. 42). A *diverging scaling function* of the polymer theory of the form

$$\mathcal{X}(z, \varepsilon) = 1 + z \left(\frac{A}{\varepsilon} + A' \right) + z^2 \left(\frac{B}{\varepsilon^2} + \frac{B'}{\varepsilon} \right) + \dots \tag{5.12}$$

has the asymptotic behavior

$$\mathcal{X}(z, \varepsilon) \underset{z \rightarrow \infty}{=} \mathcal{A}(\varepsilon) z^{(2/\varepsilon)\sigma(\varepsilon)} \sim S^{\sigma(\varepsilon)} \tag{5.13}$$

where $\mathcal{A}(\varepsilon)$ is a calculable amplitude, and where σ is the universal critical exponent at the fixed point [Eq. (3.12ter)]

$$\sigma(\varepsilon) = \frac{\varepsilon}{8} \frac{A}{2} + \left(\frac{\varepsilon}{8} \right)^2 \left(\frac{17}{8} A + 8A' + B' - AA' \right) + O(\varepsilon^3) \tag{5.14}$$

Of course, $\mathcal{X}(z, \varepsilon)$ must have a scaling behavior, and hence must be “renormalizable,” and, to this order, the precise condition for its Taylor–Laurent coefficients reads⁽⁴²⁾

$$8A + 2B - A^2 = 0 \tag{5.15}$$

So we see that this gives the complete answer to our problem.

It is first a matter of straightforward algebra to check that the coefficients A_L, A'_L, B_L, B'_L in (5.7)–(5.10) satisfy (5.15) for any L , and hence $\mathcal{Z}(\mathcal{S}_L)$ is proven (at least at this order) to have an anomalous “renormalizable” scaling behavior (5.13), with an exponent [Eq. (5.14)]

$$\gamma_L(\varepsilon) - 1 = \sigma(\varepsilon) \tag{5.16}$$

Second, inserting the values (5.7)–(5.10) into (5.14), we find after some trivial algebra the γ exponent for the star \mathcal{S}_L to $O(\varepsilon^2)$:

$$\gamma_{\mathcal{S}_L} - 1 \equiv \gamma_L - 1 = \frac{\varepsilon}{8}(3 - L)\frac{L}{2} + \left(\frac{\varepsilon}{8}\right)^2 \frac{L}{8}(8L^2 - 37L + 55) + O(\varepsilon^3) \tag{5.17}$$

5.2.1. Vertex Exponents σ_L . Now, the irreducible core exponent σ_L in (3.30) is readily obtained from (3.31):

$$\begin{aligned} \sigma_L &= \gamma_{\mathcal{S}_L} - 1 - \frac{L}{2}(\gamma_{\mathcal{S}_1} - 1) \\ &= \frac{\varepsilon}{8}(2 - L)\frac{L}{2} + \left(\frac{\varepsilon}{8}\right)^2 \frac{L}{8}(L - 2)(8L - 21) + O(\varepsilon^3) \end{aligned} \tag{5.18}$$

Inserted into Eq. (3.34) the values (5.18) give the exponent $\gamma_{\mathcal{G}}$ of any polymer network \mathcal{G} to order $O(\varepsilon^2)$.

The exponents Δ_L of the equivalent formula (3.34bis) have an $O(\varepsilon^2)$ expansion which can be derived from (5.18) and (1.3):

$$\Delta_L = -\frac{1}{4}d(L - 2) + \frac{\varepsilon}{8}\frac{1}{2}(4 - L^2) + \left(\frac{\varepsilon}{8}\right)^2 (L - 2)\left(L^2 - \frac{21}{8}L - \frac{7}{4}\right) \tag{5.19}$$

5.2.2. Scaling Dimensions x_L . The values of the dimensions x_L are of importance since they can be used in direct space [Eq. (2.40)]. They are immediately obtained from the basic relation (3.53):

$$x'_L = \frac{\varepsilon}{8}L(L - 1) + \left(\frac{\varepsilon}{8}\right)^2 \frac{L}{4}(-8L^2 + 33L - 23) + O(\varepsilon^3) \tag{5.20}$$

with

$$x_L \equiv \frac{L}{2}(d-2) + x'_L$$

Recall that these anomalous dimensions govern the critical correlation function of L infinite chains tied together

$$G_L(\mathbf{X} - \mathbf{Y}, T_c) \sim |\mathbf{X} - \mathbf{Y}|^{-L(d-2) - 2x'_L}$$

5.3. Relation to Known Exponents

Let us first relate our $O(\varepsilon^2)$ results (5.17)–(5.20) to known results. Essentially the one-chain $L=1$ case corresponds to the spin–spin correlation function of the $O(n \rightarrow 0)$ model, and the two-chain $L=2$ case to the energy–energy one [Eq. (2.40)]. We check indeed from (5.17) that

$$\gamma_{\mathcal{S}_1} - 1 = \gamma_{\mathcal{S}_2} - 1 = \frac{\varepsilon}{8} + \frac{13}{4} \left(\frac{\varepsilon}{8}\right)^2 + O(\varepsilon^3)$$

for the 1- and 2-star exponents, in agreement with the identity $\gamma_{\mathcal{S}_1} - 1 \equiv \gamma_{\mathcal{S}_2} \equiv \gamma - 1$ [Eq. (1.3)]. For the vertex exponents σ_L , we have, in agreement with our general formalism (Section 3), $2\sigma_1 = \gamma - 1$, while the two-leg exponent σ_2 satisfies (3.55),

$$\sigma_2 = 0$$

The x'_L exponents for $L=1$, $L=2$ read

$$\begin{aligned} x'_1 &= \frac{1}{2}(\varepsilon/8)^2 + O(\varepsilon^3) \\ x'_2 &= (\varepsilon/8)2 + (\varepsilon/8)^2 \quad (11/2) \end{aligned}$$

and yield through the identities $x'_1 = \eta/2$ and $2 - x'_2 = 1/\nu$,

$$\begin{aligned} \eta &= \left(\frac{\varepsilon}{8}\right)^2 + O(\varepsilon^3) \\ \nu &= \frac{1}{2} \left[1 + \frac{\varepsilon}{8} + \frac{15}{4} \left(\frac{\varepsilon}{8}\right)^2 + O(\varepsilon^3) \right] \end{aligned}$$

which are the $n=0$ values.⁽¹⁵⁾

It is also interesting to compare the ε -expansion results to the exact values in two dimensions. They are, for the star polymers,⁽¹⁾

$$\gamma_{\mathcal{S}_L} - 1 = \frac{4 + 9L(3 - L)}{64} \quad (5.21)$$

and for the vertex exponents,⁽¹⁾

$$\sigma_L = \frac{1}{64} (2-L)(9L+2) \quad (5.22)$$

and for the x_L ($\equiv x'_L$ in 2D) [Eq. (4.18)]

$$x_L = \frac{1}{48} (9L^2 - 4) \quad (5.23)$$

Actually, in 2D the x_L values are calculated exactly in the Coulomb-gas formalism, and the σ_L are derived using the general formalism described here, which is the reverse of the ε calculations presented here.

The 2D formulas are polynomials of second order in L , while ε expansions (5.17)–(5.20) involve terms $\varepsilon^n P_{n+1}(L)$, where P_{n+1} is a polynomial of order $n+1$ in L , which blows up with L , the real expansion parameter being rather εL . So this highly diverging ε expansion, which is only asymptotic, must arrange in such a way to reproduce for $\varepsilon=2$ simple quadratic polynomials (5.21)–(5.23)!

Another striking feature is the analytic continuation of L to $L=0$. For trivial and purely combinatorial reasons, a factor L (i.e., the number of choices of the first arm in \mathcal{S}_L) always appears in $\gamma_{\mathcal{S}_L}$, σ_L , or x_L [(5.17)–(5.20)] to *all* orders in ε . Hence, in the ε expansion, $\gamma_{\mathcal{S}_{L \rightarrow 0}} = 1$, $\sigma_{L \rightarrow 0} = 0$, $x_{L \rightarrow 0} = 0$, order by order in ε . However, the extension of exact results (5.21)–(5.23) to $L \rightarrow 0$ gives in 2D

$$\gamma_{\mathcal{S}_0} - 1 = 1/16, \quad \sigma_0 = 1/16, \quad x_0 = -1/12 \quad (5.24)$$

So the two limits $L \rightarrow 0$, $\varepsilon \rightarrow 2$ do not commute. The two-dimensional nontrivial results are related to the anomaly or central charge c in 2D. Indeed, we see from (4.17) and (4.18) that $x_0 = 2h_{0,0} = -1/4m(m+1)$ for $m=2$, while c is given by (4.16). It can be shown (by extending the arguments of refs. 41, 16, and 17) that $x_0 = -1/12$ could correspond to the repulsion between two sets of concentric polymer rings located around two points **X** and **Y**, respectively. The number of rings around each point is arbitrarily large and the mean lengths are infinite. So a *zero-star* in *two dimensions* should be seen as a system of *successive rings entangled with a point (the core)*. The nontrivial values (5.24) are the reflection of this “anomaly.” Polymer lines can be entangled with a point only in 2D, and not in higher dimension. Hence, this effect disappears in $d > 2$, and, of course, cannot be seen in the ε expansion.

5.4. Contact Exponents

From the knowledge of the σ_L or the x_L one can derive other interesting critical exponents, namely the *contact exponents*. These exponents were introduced by des Cloizeaux⁽⁹⁾ for the contact probability between points of a single chain. More precisely, one has

$$P_a(r) = R^{-d} F_a(r/R), \quad a = 0, 1, 2 \quad (5.25)$$

where P_a is the probability that the two extremities of a segment inside a single chain are at a relative distance r (Fig. 3). By convention,⁽⁹⁾ $a = 0$ corresponds to the two extremities of a chain, $a = 1$ to the contact of one extremity inside the chain, and $a = 2$ to that of the interior points.

In (5.25), F_a is a universal function, which depends on the location of the considered segment along the chain, the scale R being its swollen size.^(8,9) The universal *contact exponents* arise when one lets $r \rightarrow 0$. Then

$$F_a(x) \underset{x \rightarrow 0}{\sim} x^{\theta_a} \quad (5.26)$$

The contact exponent θ_0 of the two extremities is well known to be related to the usual γ exponent of the chain by⁽⁹⁾

$$\theta_0 = \frac{\gamma - 1}{\nu}$$

Incidentally, this shows again that the γ exponent is really an *end effect* in polymer theory,^(9,1) in agreement with the general formula (3.35). In the theory of self-avoiding tethered manifolds,⁽⁴⁹⁻⁵¹⁾ which generalizes the polymer theory, this leads to a nonuniversal and shape-dependent γ .^(51,49) Note also that the full universal functions F_a can be computed explicitly to first order in ε .⁽⁸⁾

Many more higher-order contact exponents can be defined,⁽¹¹⁾ as alluded to in the introduction. I have described their physics and derived the scaling relations between them in detail in refs. 11 and 13. Here I only repeat the essential findings.

First one can ask about the probability of forming a certain set of contacts inside a linear polymer chain⁽¹¹⁾ by folding the latter onto itself. Then one obtains a graph \mathcal{G} of contacts (Fig. 9). The probability of forming such a topology is of course $P_{\mathcal{G}} = \mathcal{Z}(\mathcal{G})/\mathcal{Z}(\mathcal{S}_1)$, where $\mathcal{Z}(\mathcal{G})$ is the restricted partition function of the chain with fixed vertices forming the contacts \mathcal{G} , i.e., just the partition function of the graph \mathcal{G} as in (2.2). $\mathcal{Z}(\mathcal{S}_1)$ is similarly just the full partition function of the single chain. Hence we find⁽¹¹⁾

$$P_{\mathcal{G}} \sim S^{\gamma_{\mathcal{G}} - \gamma} \sim S^{-\chi_{\mathcal{G}}}$$

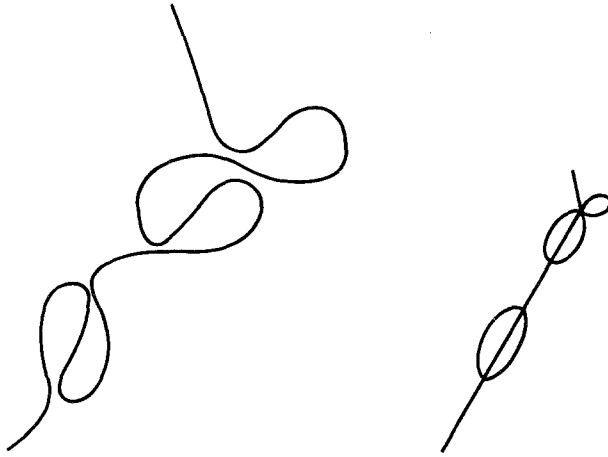


Fig. 9. Self-contacts of a single chain and the associated graph \mathcal{G} of contacts. The probability for a chain to form the \mathcal{G} contact is $P_{\mathcal{G}} \sim S^{-\gamma_{\mathcal{G}}}$ with $\gamma_{\mathcal{G}} \equiv \gamma - \gamma_{\mathcal{G}}$.

where

$$\gamma_{\mathcal{G}} = \gamma - \gamma_{\mathcal{G}} \tag{5.27}$$

is the \mathcal{G} -contact exponent⁽¹¹⁾ governing the probability of forming a certain folded structure at some prescribed points along the chain (note indeed that one does not let the contact points slip along the chain; there are no slip links here). In the case of the probability of forming a ring at the head of a chain (Fig. 3a), i.e., a “tadpole,” the exponent γ_1 was called the “limiting ring closure probability” exponent in early (two-dimensional) studies.^(52,53) Using the general formulas (3.34) or (3.54), one finds the explicit and universal expression of $\gamma_{\mathcal{G}}$ in terms of the scaling dimensions of the contact vertices so formed in \mathcal{G} . Let us first apply this to the three simplest cases of contacts ($a=0$, end–end; $a=1$, end–interior; $a=2$, interior–interior) (Fig. 3). The graphs so formed are described by the sets of vertices:

$$\begin{aligned} a=0, & \quad n_L=0, & \quad \forall L \\ a=1, & \quad n_1=1, & \quad n_3=1 \\ a=2, & \quad n_1=2, & \quad n_4=1 \end{aligned} \tag{5.28}$$

and a loop number $\mathcal{L} = 1$, Hence

$$\begin{aligned} \gamma_0 &= 2\sigma_1 + vd = \gamma - 1 + vd \\ \gamma_1 &= 2\sigma_1 + vd - \sigma_1 - \sigma_3 = \sigma_1 - \sigma_3 + vd \\ \gamma_2 &= 2\sigma_1 + vd - 2\sigma_1 - \sigma_4 = vd - \sigma_4 \end{aligned}$$

Now, one establishes easily^(11,5) a well-known scaling relation between these contact exponents (5.27) describing the scaling in terms of the size of the chain, and the contact exponents $\theta_0, \theta_1, \theta_2$ describing the approach in direct space. It is $\gamma_a = \nu(d + \theta_a)$.⁽¹¹⁾ Hence we find explicitly

$$\begin{aligned} \nu\theta_0 &= 2\sigma_1 \\ \nu\theta_1 &= \sigma_1 - \sigma_3 \\ \nu\theta_2 &= -\sigma_4 \end{aligned} \quad (5.29)$$

Note that the θ exponents depend only on the *local* scaling powers σ of the vertices upon contact. Indeed, the relations (5.29) can be written, since $\sigma_2 = 0$,

$$\nu\theta_0 = 2\sigma_1 - \sigma_2, \quad \nu\theta_1 = \sigma_1 + \sigma_2 - \sigma_3, \quad \nu\theta_2 = 2\sigma_2 - \sigma_4$$

which is just the general mnemonic subtraction rule (1.7) mentioned in the introduction.

I now give the full generalization which describes any type of contact for a polymeric system. It will embrace all cases, namely successive contacts of any order inside a linear chain,⁽¹¹⁾ contacts between branching vertices inside a network,⁽¹³⁾ approach exponents between the cores of several stars, etc. Let us consider a set of I polymer vertices with respective number of lines $L_i, i = 1, \dots, I$, at positions \mathbf{p}_i in d -dimensional space (Fig. 10). Notice that these vertices can be vertices of the same connected network, or they can belong to several (disconnected) networks, e.g., several stars. In the first case, the probability of approach that we consider is measured relative to the total set of configurations of the embedding network, while

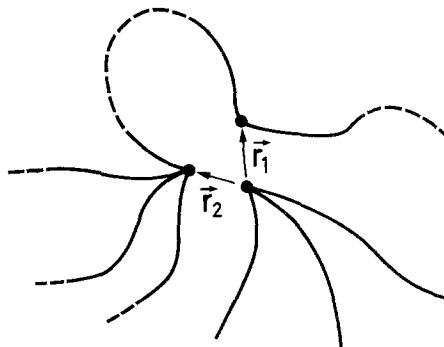


Fig. 10. Contact of several branching points inside a network, or contact of several stars.

The contact exponent is $\theta_{\{L_i\}} = x_{\sum_i L_i} - \sum_i x_{L_i}$.

in the second case, it is a *relative* probability, related to the total number of configurations of the various pieces fixed at some points far apart. The contact exponents do not depend on the connectivity of the networks. As we shall see, they are purely *local* and given by rules very similar to those of an operator-product expansion in field theory.

Now, our I vertices at positions $\{\rho_i\}$ define a set of $I-1$ independent relative positions $= \{\mathbf{r}_i = \rho_i - \rho_I, i = 1, \dots, I-1\}$, sufficient to describe this geometry of contact. Let $P\{\mathbf{r}_i, i = 1, \dots, I-1\}$ be its probability. Suppose now that all these relative positions go to zero together, with $\mathbf{r}_i = x\mathbf{u}_i$, $x \rightarrow 0$, and the vectors \mathbf{u}_i being of fixed length. Then

$$P\{\mathbf{r}_i = x\mathbf{u}_i\} \underset{x \rightarrow 0}{\sim} x^{\theta_{\{L_i\}}} \tag{5.30}$$

where $\theta_{\{L_i\}}$ is the contact exponent of the set of vertices $\{L_i\}$. It has an expression in terms of the scaling dimensions σ_{L_i} or x_{L_i} of the vertices upon contact, which can be found^(11,13) by easy scaling analysis and use of the fundamental additive law (3.34) giving $\gamma_{\mathcal{G}}$ for a network \mathcal{G} . One considers indeed the $\{L_i\}$ vertices as being embedded in a larger connected network \mathcal{G} (there is an infinite freedom in choosing such a \mathcal{G}). Then one can relate the $\theta_{\{L_i\}}$ to the difference between the $\gamma_{\mathcal{G}}$ exponent of the network before the contact and to the $\gamma_{\mathcal{G}_c}$ of the new network \mathcal{G}_c upon contact. The details can be found in Section 5 of ref. 13. One finds an exponent $\theta_{\{L_i\}}$, which is independent of the choice of \mathcal{G} (as it must) and depends only on the $\{L_i\}$ through

$$v\theta_{\{L_i\}} = \sum_i \sigma_{L_i} - \sigma_{\Sigma_i L_i} \tag{5.31}$$

The interpretation is illuminating: one compares the total scaling dimension before contact to that generated by the fusion of the L_i vertices upon contact. This result applies to any number of vertices upon contact. It contains as a particular case Eq. (1.7) for the contact exponent $\theta_{L,L'}$ of the cores of two L - and L' -arm stars. In terms of the exponents x_{L_i} , the relation is [use (3.53)]

$$\theta_{\{L_i\}} = x_{\Sigma_i L_i} - \sum_i x_{L_i} \tag{5.32}$$

Note that by linearity in $\{L_i\}$ this formula holds for the *total* scaling dimensions x_L [(4.14)ter] *as well as* for the *anomalous parts* x'_L . Now, results (5.18) or (5.20) give all these contact exponents to order $O(\varepsilon^2)$. I illustrate it in some interesting cases. First, let us check the values obtained in case (5.26), (5.28) by des Cloizeaux.⁽⁹⁾ We have according to (5.20) at order ε^2

$$\begin{aligned}
 x'_1 &= \frac{1}{2} \left(\frac{\varepsilon}{8} \right)^2 = \frac{\eta}{2}, & x'_2 &= \frac{\varepsilon}{8} 2 + \left(\frac{\varepsilon}{8} \right)^2 \frac{11}{2} \\
 x'_3 &= \frac{\varepsilon}{8} 6 + \left(\frac{\varepsilon}{8} \right)^2 3, & x'_4 &= \frac{\varepsilon}{8} 12 + \left(\frac{\varepsilon}{8} \right)^2 (-19)
 \end{aligned}
 \tag{5.33}$$

Hence we find

$$\begin{aligned}
 \theta_0 &= x_2 - 2x_1 = \frac{\varepsilon}{4} + \frac{9}{128} \varepsilon^2 \\
 \theta_1 &= x_3 - x_2 - x_1 = \frac{\varepsilon}{2} - \frac{3}{64} \varepsilon^2 \\
 \theta_2 &= x_4 - 2x_2 = \varepsilon - \frac{15}{32} \varepsilon^2
 \end{aligned}
 \tag{5.34}$$

in perfect agreement with the field-theoretic values of ref. 9. This is an independent verification of the latter, since they were published in 1980. It can also be considered as an indirect *check* of the general *multiplicative renormalization* structure given in Section 3. Note also that when the degree of contact increases, larger coefficients appear in front of $\varepsilon/8$ or $(\varepsilon/8)^2$ in (5.34), and this is due to the effective εL expansion in (5.20). Hence the ε expansion is highly diverging and resummation procedures should be used, the simplest one being a Padé.⁽⁹⁾

Another interesting case is the exponent for the *three-body* contacts inside a chain, which plays a dominant role at the tricritical Θ point. It corresponds to a set $\{L_{i=1,2,3}\} = (2, 2, 2)$. Hence we find the new value from (5.31), (5.32),

$$\theta_{\text{three-body}} = x_6 - 3x_2 = -\sigma_6/\nu \tag{5.35}$$

In $d = 4 - \varepsilon$ dimensions, using (5.18), we have the new ε^2 result

$$\theta_{\text{three-body}} = 3\varepsilon - (93/32) \varepsilon^2 \tag{5.36}$$

In 2D, due to (5.22)⁽¹⁾ and $\nu = 3/4$,⁽¹⁶⁾ the *exact* value is

$$\theta_{\text{three-body}} = 14/3 \tag{5.37}$$

Finally, we are now able to give the contact exponents (1.7) $\theta_{L,L'}$ of two L, L' -arm stars, described in the introduction (Fig. 1). From (5.32) they read simply

$$\theta_{L,L'} = x_{L+L'} - x_L - x_{L'} \tag{5.38}$$

The values (5.20) of x_L give the $O(\varepsilon^2)$ expansion

$$\theta_{L,L'} = 2LL' \left\{ \frac{\varepsilon}{8} + \left(\frac{\varepsilon}{8} \right)^2 \left[\frac{33}{4} - 3(L + L') \right] + O(\varepsilon^3) \right\} \quad (5.39)$$

valid for any $L, L' \geq 1$. In particular, we recover the above standard exponents (5.34), $\theta_0 \equiv \theta_{1,1}$, $\theta_1 \equiv \theta_{1,2}$, $\theta_2 = \theta_{2,2}$ (Fig. 3).

In two dimensions we know the exact values of irreducible exponents σ_L , (5.22), or x_L , (5.23). Hence, the star-contact exponents have the exact values

$$\theta_{L,L'} = (1/24)(9LL' + 2) \quad (5.40)$$

The usual exponents $\theta_0, \theta_1, \theta_2$ are then⁽¹¹⁾

$$\begin{aligned} \theta_0 = \theta_{1,1} &= 11/24 = (\gamma - 1)/\nu \\ \theta_1 = \theta_{1,2} &= 5/6, \quad \theta_2 = \theta_{2,2} = 19/12 \end{aligned} \quad (5.41)$$

Note that again the analytic structures of the ε expansion (5.39) of $\theta_{L,L'}$ and of the exact 2D value (5.40) are quite different. In particular, for the contact of an L -leg star with a *zero*-star, $\theta_{L,0} \equiv 0$ in the ε expansion (to all orders), while in 2D it develops a nontrivial value

$$\theta_{L,0} = 1/12 \quad (5.42)$$

which reflects the nontrivial zero-star of the above.

6. SURFACE EXPONENTS

We can generalize the previous study of polymer networks in a good solvent to that of networks grafted onto a *surface*. This was achieved in ref. 22 in the *two-dimensional* case for networks near a repelling surface, i.e., at the so-called *ordinary surface transition*^(23,54-60) (see ref. 58 for a review). In ref. 22, I gave the exact value in 2D of the exponent $\gamma_{\mathcal{G}}^S$ of any network attached to the surface. However, the surface scaling analysis given there (similar to that of Section 4 above) was quite general. It applies equally well in d dimensions, and I shall describe it in this paper. I study here only the *ordinary* surface transition. The special transition,^(23,61,62) where the polymers just adsorb, could be studied in the same way.

6.1. Networks near a Surface

Consider a general polymer network \mathcal{G} made up of \mathcal{N} identical long chains of length S . Some of the vertices belong to the surface (or actually

stay *near* the surface, the latter being infinitely repulsive at the ordinary transition, i.e., for infinite polymers). Then topological characteristics of the network are the numbers $\{n_L, L \geq 1\}$ of L -leg vertices floating in the bulk solvent and the numbers⁽²²⁾ $\{n_L^S, L \geq 1\}$ of L -leg vertices near (or “grafted” onto) the surface (Fig. 11). In terms of these, the total number \mathcal{N} of chains, the numbers of vertices \mathcal{V} in the bulk and \mathcal{V}_S on the surface, and the number \mathcal{L} of constitutive polymer loops are given by, respectively,

$$\begin{aligned} \mathcal{N} &= \sum_{L \geq 1} \frac{1}{2} L(n_L + n_L^S) \\ \mathcal{V} &= \sum_{L \geq 1} n_L, \quad \mathcal{V}_S = \sum_{L \geq 1} n_L^S \\ \mathcal{L} &= \frac{1}{2} \sum_{L \geq 1} (L-2)(n_L + n_L^S) + 1 \end{aligned} \quad (6.1)$$

All the possible geometrical configurations are obtained by our *moving* all surface vertices (except one fixed to eliminate translational invariance) *along* the surface, and by moving all the bulk vertices in the half-space.⁽²²⁾

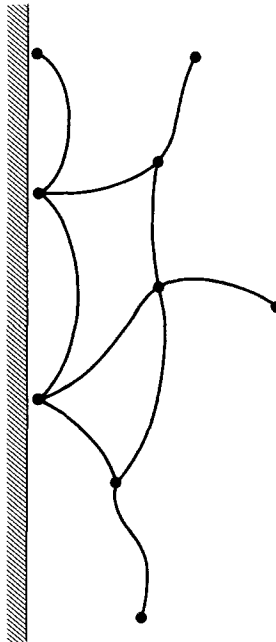


Fig. 11. A network \mathcal{G} grafted onto a surface. One surface vertex is fixed, the other ones are moved freely when calculating $\mathcal{L}^S(\mathcal{G})$.

Then the partition function of the network so constrained in a restricted geometry will scale like

$$\mathcal{Z}^S(\mathcal{G}) \sim \mu^{\mathcal{N}^S} S^{\gamma_{\mathcal{G}}^S - 1}, \quad S \rightarrow \infty \tag{6.2}$$

where $\gamma_{\mathcal{G}}^S$ is a new surface-bulk exponent. μ is still of course⁽⁶³⁾ the same connectivity constant as in the bulk. In ref. 22, I gave the exact value of $\gamma_{\mathcal{G}}^S$ in 2D for \mathcal{G} described by $\{n_L, n_L^S\}$:

$$\gamma_{\mathcal{G}}^S = \frac{1}{4} + \frac{1}{64} \sum_{L \geq 1} n_L (2-L)(9L+50) - \frac{1}{32} \sum_{L \geq 1} n_L^S (9L^2 + 22L - 24) \tag{6.3}$$

The bulk terms with n_L are the same as the contributions in the usual bulk exponents⁽¹⁾ $\gamma_{\mathcal{G}}$ described in the above sections. The surface contributions with the n_L^S are new and show that, here again, each vertex on the surface yields its own contribution. The 2D formula (6.3) is absolutely general; it agrees with the exact results by Cardy⁽⁶⁰⁾ for the one-chain exponents $\gamma_1 = 61/64$ and $\gamma_{11} = -3/16$, and with their numerical counterparts.^(64,65) Equation (6.3) has been beautifully checked numerically by Colby *et al.*⁽⁶⁶⁾ for various star polymers near a surface line or in a wedge geometry⁽²²⁾ in two dimensions.

The derivation of Eq. (6.3) given in ref. 22 is completely analogous to that given in Section 4 here, and introduced in addition in the associated $O(n)$ field theory surface vertex operators ϕ_L^S , with their own surface scaling dimensions at criticality,

$$G_L^S(\mathbf{X} - \mathbf{Y}, T_c) = \langle \phi_L^S(\mathbf{X}) \phi_L^S(\mathbf{Y}) \rangle_{T_c} = |\mathbf{X} - \mathbf{Y}|^{-2x_L^S} \tag{6.4}$$

As in Eqs. (2.36), (2.40), and (4.2), G_L^S is the polydisperse correlation function of L polymer chains, with fluctuating lengths, and tied together at vertices \mathbf{X} and \mathbf{Y} , now very close to the surface (Fig. 12). In terms of these dimensions x_L^S and of the bulk ones x_L of (4.2), the exponent $\gamma_{\mathcal{G}}^S$ was shown to read

$$\gamma_{\mathcal{G}}^S - 1 = \nu \left[2\mathcal{V} + \mathcal{V}_S - 1 - \sum_{L \geq 1} (n_L x_L + n_L^S x_L^S) \right] - \mathcal{N} \tag{6.5}$$

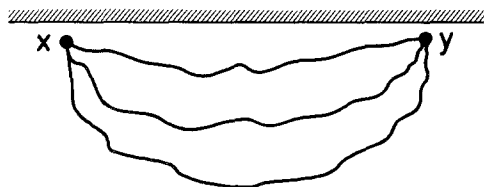


Fig. 12. A watermelon network grafted along the surface. Its polydisperse correlation function G_L^S at criticality scales like $G_L^S(\mathbf{X} - \mathbf{Y}) = |\mathbf{X} - \mathbf{Y}|^{-2x_L^S}$.

a formula generalizing the bulk result in two dimensions (4.14). It is not difficult to show, by following the arguments of Section 4 or of ref. 22, that in d dimensions one has simply to replace $2\mathcal{V}$ by $d\mathcal{V}$ and $\mathcal{V}_S - 1$ by $(d-1)(\mathcal{V}_S - 1)$ as in (4.14bis), and x_L of (4.14ter) is the total scaling dimension $x_L = L(d-2)/2 + x'_L$ appearing in (2.40). Hence, the exact (hyper)scaling relation is in d dimensions

$$\gamma_{\mathcal{G}}^S - 1 = \nu \left[d\mathcal{V} + (d-1)(\mathcal{V}_S - 1) - \sum_{L \geq 1} (n_L x_L + n_L^S x_L^S) \right] - \mathcal{N} \quad (6.6)$$

In two dimensions, the exponents x_L^S have been derived⁽²²⁾ by strip numerical calculations and using conformal invariance:

$$x_L^S = h_{L+1,1} = L(3L+2)/8 \quad (6.7)$$

Inserting these values and the bulk exponents x_L of (5.23) into (6.5) and using the topological relations (6.1), one finds the exact formula (6.3).

In d dimensions, I first derive the general formula (6.6) by another method, devising a direct renormalization for polymer networks near a surface. Then I calculate the x_L^S surface exponents to $O(\varepsilon)$, by considering simple star polymers near a surface. One finds

$$x_L^S = \frac{L}{2} d + x'_L{}^S \quad (6.8)$$

where $Ld/2$ is the Brownian value, and $x'_L{}^S$ is the anomalous part generated by self-avoidance in $d = 4 - \varepsilon$ dimensions,

$$x'_L{}^S = \frac{\varepsilon}{8} L(L-2) + O(\varepsilon^2) \quad (6.9)$$

Note that in two dimensions the anomalous part is exactly, from (6.7),

$$x'_L{}^S = x_L^S - L = \frac{3}{8} L(L-2) \quad (6.10)$$

exhibiting an (*exact*) quadratic form strikingly similar to the $O(\varepsilon)$ result! [See ref. 11 for the discussion of a similar resemblance between $O(\varepsilon)$ and exact 2D results in the bulk.] The value (6.8), (6.9), together with the bulk result

$$x_L = \frac{L}{2} (d-2) + \frac{\varepsilon}{8} L(L-1) \quad (6.11)$$

give the complete $\gamma_{\mathcal{G}}^S$ surface exponent (6.6) to first order in ε .

Let us now derive the $O(\varepsilon)$ results (6.8), (6.9) by a direct method. For this we first need to analyze the Brownian behavior of a network near a surface.

6.2. Brownian Networks

6.2.1. Single Chain near a Surface. To build the partition function of “phantom” Brownian networks near a surface, one first requires that of a single Brownian chain. We consider a d -dimensional half-space in the presence of a $d-1$ hypersurface, the position of a point being labeled by (\mathbf{r}, z) , with $\mathbf{r} \in \mathbb{R}^{d-1}$ and $z > 0$ as the distance to the hyperplane. Then the Brownian partition function of a chain with fixed extremities at \mathbf{r}, z and \mathbf{r}', z' (Fig. 13) is defined as the functional integral

$$\begin{aligned}
 G_0^S(\mathbf{r}, z, \mathbf{r}', z') = & \int d^+ \{\mathbf{r}\} \delta^{d-1}(\mathbf{r}_{||}(0) - \mathbf{r}) \\
 & \times \delta^{d-1}(\mathbf{r}_{||}(S) - \mathbf{r}') \delta(z(0) - z) \delta(z(S) - z') P_0\{\mathbf{r}\} \\
 & \times \left(\int d\{\mathbf{r}\} P_0\{\mathbf{r}\} \delta^d[\mathbf{r}(0)] \right)^{-1} \tag{6.12}
 \end{aligned}$$

where $\mathbf{r}(s)$ is the complete configuration in d space of the chain

$$\mathbf{r}(s) = (\mathbf{r}_{||}(s), z(s))$$

and where P_0 is the pure Brownian weight:

$$P_0\{\mathbf{r}\} = \exp \left\{ -\frac{1}{2} \int_0^S ds [\dot{\mathbf{r}}_{||}^2(s) + \dot{z}^2(s)] \right\}$$

In (6.12) the measure for functional integration is

$$\int d^+ \{\mathbf{r}\} \equiv \int_{\mathbb{R}^{d-1}} d\{\mathbf{r}_{||}\} \int_{z>0} d\{z\} \exp \left[-\int_0^S \mathcal{U}(z(s)) ds \right]$$

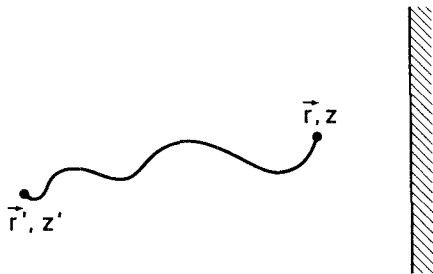


Fig. 13. Labeling of the positions of the extremities of a single chain near a $(d-1)$ -dimensional wall.

where \mathcal{U} measures the potential of the surface, and where the configurations are restricted to the $z > 0$ half-space, the interaction being infinitely repulsive for $z < 0$.

It is now well known^(67,23) that for long chains the effect of the surface interaction can be modeled by a single surface contact parameter c such that the correlator (6.12) obeys the boundary condition

$$\left. \frac{\partial G_0^S}{\partial z} \right|_{z=0} = c G_0^S|_{z=0} \quad (6.13)$$

The value of c depends on \mathcal{U} near the boundary. I describe here the *repelling case*, i.e., the ordinary surface transition⁽²³⁾ where the chain cannot touch the surface. It happens, for instance, for a potential \mathcal{U} which is $+\infty$ for $z < 0$ and 0 for $z > 0$, with no (sufficiently) attractive part near $z = 0$. This corresponds to the limit $c \rightarrow +\infty$, and to a *Dirichlet boundary condition*^(68,23)

$$G_0^S|_{z=0 \text{ or } z'=0} = 0 \quad (6.14)$$

[$c = 0$ corresponds to a Neumann boundary condition, and is the Brownian value of c for the special transition where a (Brownian) chain just adsorbs]. This Brownian correlator (6.12) is well known (see, in particular, Lépine and Caillet⁽⁶⁸⁾) and reads in the Dirichlet case

$$G_0^S(\mathbf{r}, z, \mathbf{r}', z') = (2\pi S)^{-d/2} \exp\left[-\frac{1}{2S}(\mathbf{r}-\mathbf{r}')^2\right] \\ \times \left\{ \exp\left[-\frac{1}{2S}(z-z')^2\right] - \exp\left[-\frac{1}{2S}(z+z')^2\right] \right\} \quad (6.15)$$

satisfying (6.14). The minus sign between the two *image* terms in (6.15) is characteristic of Dirichlet boundary conditions. For Neumann ones the correlator is simply the symmetric one

$$G_{\text{Neumann}}^S(\mathbf{r}, z, \mathbf{r}', z') = (2\pi S)^{-d/2} \exp\left[-\frac{1}{2S}(\mathbf{r}-\mathbf{r}')^2\right] \\ \times \left\{ \exp\left[-\frac{1}{2S}(z-z')^2\right] + \exp\left[-\frac{1}{2S}(z+z')^2\right] \right\}$$

Note also that when one of the extremities goes far away from the interface, $z \rightarrow \infty$ or $z' \rightarrow \infty$, (6.12) recovers the bulk partition (2.22), (2.23), the second image term dropping away.

In the case (6.15) of the ordinary surface transition the following integration formula will be useful:

$$\begin{aligned} \mathcal{Z}_0(z, S) &\equiv \int_0^\infty dz' \int d^{d-1}r' G_0^S(\mathbf{r}, z, \mathbf{r}', z') \\ &= 2(2\pi S)^{-1/2} \int_0^z dx e^{-x^2/2S} \end{aligned} \quad (6.16)$$

and represents the total (normalized) partition function of a chain with one extremity fixed at (\mathbf{r}, z) . In agreement with the normalization by the bulk Brownian partition function chosen in (6.12), we see that far away from the surface

$$\mathcal{Z}_0(z \rightarrow \infty, S) = 1$$

while when the extremity of the chain approaches the surface, it vanishes like

$$\mathcal{Z}_0(z \rightarrow 0, S) \approx 2(2\pi S)^{-1/2} z \quad (6.17)$$

due to the strong repulsion of the surface. So we cannot define directly the partition function of a chain attached by one extremity to the surface. A good way to define a similar quantity is to consider the *normal derivative*

$$\begin{aligned} &\left. \frac{\partial}{\partial z} G_0^S(\mathbf{r}, z, \mathbf{r}', z') \right|_{z=0} \\ &= \frac{2z'}{S} (2\pi S)^{-d/2} \exp\left[-\frac{1}{2S}(\mathbf{r}-\mathbf{r}')^2\right] \exp\left(-\frac{1}{2S}z'^2\right) \end{aligned} \quad (6.18)$$

which stays finite at the surface. (A similar trick is used in the field-theoretic study of the ordinary surface transition, where the surface field vanishes, while its normal derivative does not.^(56,59))

We so define a total Brownian partition function for a chain upon contact to the interface

$$\begin{aligned} \partial_\perp \mathcal{Z}_0(S) &\equiv \int d^{d-1}r' \int_0^\infty dz' \left. \frac{\partial}{\partial z} G_0^S(\mathbf{r}, z, \mathbf{r}', z') \right|_{z=0} \\ &= \left. \frac{\partial}{\partial z} \mathcal{Z}_0(z, S) \right|_{z=0} \end{aligned} \quad (6.19)$$

which replaces (6.16). It is readily calculated from (6.17)

$$\mathcal{Z}_1^B(S) \equiv \partial_\perp \mathcal{Z}_0(S) = 2(2\pi S)^{-1/2} \quad (6.20)$$

and is one (differential) Brownian partition function of a chain approaching a “Dirichlet” surface. Note that according to this definition the γ_1 “magnetic” surface exponent^(22,23,58–60,64) such that

$$\mathcal{Z}_1(S) \equiv \partial_{\perp} \mathcal{Z}(S) \sim S^{\gamma_1 - 1}$$

is in the Brownian case

$$\gamma_1^{\text{B}} = 1/2 \quad (6.21)$$

If we want the usual γ_{11} exponent governing the partition function of a single chain attached by the two extremities to the wall, we simply differentiate (6.18) with respect to the other extremity z' at $z'=0$ and define

$$\mathcal{Z}_{11}(S) = (\partial_{\perp})^2 \mathcal{Z}_0 \equiv \int d^{d-1}r' \frac{\partial}{\partial z'} \frac{\partial}{\partial z} G_0^{\text{S}} \Big|_{z=0, z'=0}$$

since

$$\frac{\partial}{\partial z'} \frac{\partial}{\partial z} G_0^{\text{S}} \Big|_{z=0, z'=0} = \frac{2}{S} (2\pi S)^{-d/2} \exp \left[-\frac{1}{2S} (\mathbf{r} - \mathbf{r}')^2 \right] \quad (6.22)$$

we find

$$\mathcal{Z}_{11}^{\text{B}}(S) = \frac{2}{S} (2\pi S)^{-1/2}$$

such that

$$\gamma_{11}^{\text{B}} = -1/2 \quad (6.23)$$

Note that (6.21) and (6.23) satisfy, as they must, Barber’s scaling law⁽⁶⁹⁾

$$2\gamma_1 - \gamma_{11} = \gamma + \nu \quad (6.24)$$

since in the Brownian case $\gamma = 1$, $\nu = 1/2$.

6.2.2. Surface Contact of Several Chains. Let us now consider a network of Brownian chains approaching the surface, and, more particularly, a watermelon network \mathcal{W}_L (Fig. 12). All the chains, being Brownian, are independent, and since their individual partition functions all vanish when approaching the surface as in (6.17), we have to *differentiate L times* for each extremity in order to get a finite partition function. So we define the surface watermelon partition function

$$\mathcal{Z}_{\mathcal{W}_L}^{\text{S}}(S) = \int d^{d-1}r' \left[\frac{\partial}{\partial z} \frac{\partial}{\partial z'} G_0^{\text{S}}(\mathbf{r}, 0, \mathbf{r}', 0) \right]^L \quad (6.25)$$

Using the Brownian expression (6.22) yields immediately

$$\mathcal{Z}_{\mathcal{W}_L}^S(S) = [(2/S)(2\pi S)^{-d/2}]^L (2\pi S/L)^{(d-1)/2} \tag{6.26}$$

with a configuration surface exponent

$$\gamma_{\mathcal{W}_L}^{S,B} - 1 = -\frac{1}{2}d(L-1) - L - \frac{1}{2} \tag{6.27}$$

For $L=1$, we recover (6.23).

We can also consider the Brownian partition function of a star polymer, attached to the surface by its L -arm core (Fig. 14). We define it as

$$\begin{aligned} \mathcal{Z}_{\mathcal{S}_L}^{S,B}(S) &= \left[\int d^{d-1}r' \int_0^\infty dz' \frac{\partial}{\partial z} \Big|_{z=0} G_0^S(\mathbf{r}, 0, \mathbf{r}', \mathbf{z}') \right]^L \\ &\equiv [\mathcal{Z}_\gamma^B(S)]^L \end{aligned} \tag{6.28}$$

and find from (6.20)

$$\mathcal{Z}_{\mathcal{S}_L}^{S,B} = [2(2\pi S)^{-1/2}]^L \tag{6.29}$$

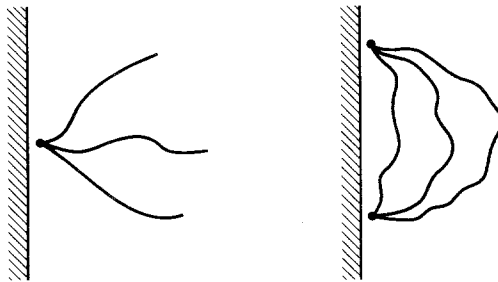
such that the surface γ exponent (6.2) reads

$$\gamma_{\mathcal{S}_L}^{S,B} - 1 = -L/2 \tag{6.30}$$

Below I derive a general surface hyperscaling law which embodies Barber's. It reads, in the presence of excluded volume,

$$2\gamma_{\mathcal{S}_L}^S - \gamma_{\mathcal{W}_L}^S = L\gamma + \nu + (L-1)(\nu d - 1) \tag{6.31}$$

and thus relates the exponents of the L -star attached to a surface and of the L watermelon (Fig. 14). For $L=1$, we just recover the famous identity



$$2\delta_{\mathcal{S}_L}^S - \delta_{\mathcal{W}_L}^S = L\delta + \nu + (L-1)(\nu d - 1)$$

Fig. 14. The generalization of Barber's surface scaling law $2\gamma_1 - \gamma_{11} = \gamma + \nu$ for $L=1$ to L -arm star polymers, $L > 1$.

(6.24). The Brownian values (6.27), (6.30) of course satisfy (6.31) for $\nu = 1/2$, $\gamma = 1$.

6.2.3. Correlator of a Brownian Surface Watermelon. It is also instructive to calculate the field-theoretic correlator (6.4) of the polydisperse watermelon network near a surface (Fig. 12) in the Brownian case. We define in general by analogy to the bulk correlator (2.36)

$$G_L^S(\mathbf{X} - \mathbf{Y}, T) = \int_0^\infty \prod_{a=1}^L dS_a e^{-T(S_1 + \dots + S_L)} \mathcal{Z}_{\mathcal{W}_L}^S(\mathbf{X} - \mathbf{Y}; S_1, \dots, S_L) \quad (6.32)$$

where the watermelon is now polydisperse with lengths S_1, \dots, S_L , and tied to the wall at \mathbf{X} and \mathbf{Y} . In the Brownian case, the polydisperse partition function $\mathcal{Z}_{\mathcal{W}_L}^S$ reads, in complete analogy to (6.25),

$$\begin{aligned} \mathcal{Z}_{\mathcal{W}_L}^{S,B}(\mathbf{X} - \mathbf{Y}; S_1, \dots, S_L) &= \prod_{a=1}^L \frac{\partial}{\partial z} \frac{\partial}{\partial z'} G_0^S(\mathbf{X}, 0, \mathbf{Y}, 0; S_a) \\ &= \prod_{a=1}^L \frac{2}{S_a} (2\pi S_a)^{-d/2} \exp \left[- \sum_{a=1}^L \frac{1}{2S_a} (\mathbf{X} - \mathbf{Y})^2 \right] \end{aligned} \quad (6.33)$$

Hence the critical correlator is, at the Brownian critical temperature $T_c = 0$,

$$G_L^{S,B}(\mathbf{X} - \mathbf{Y}, T_c = 0) = [(2\pi)^{-d/2} 2^{1+d/2} \Gamma(d/2)]^L |\mathbf{X} - \mathbf{Y}|^{-Ld} \quad (6.34)$$

It is of the expected form (6.4), with a Brownian critical scaling dimension (6.8),

$$x_L^{S,B} = Ld/2 \quad (6.35)$$

It is now easy to generalize the above findings to any Brownian network near a Dirichlet surface.

6.2.4. Surface Scaling of a Brownian Network. We have seen in Section 2.3, Eq. (2.15), that a monodisperse Brownian network in the *bulk* has an exact partition function

$$\mathcal{Z}^B(\mathcal{G}) = (2\pi S)^{-\mathcal{L}d/2} (\det C_{\mathcal{G}/\{a\}})^{-d/2} \quad (6.36)$$

where \mathcal{L} is the number (2.13) of independent loops of \mathcal{G} .

If we fix now the positions \mathbf{R}_i of all the vertices $i = 1, \dots, \mathcal{V}_B$ in bulk space as in (2.3), we are certain to have a reduced partition function of the scaling form

$$\begin{aligned} \mathcal{Z}^B\{\mathcal{G}, \mathbf{R}_i, i = 1, \dots, \mathcal{V}_B\} &= (2\pi S)^{-\mathcal{L}d/2 - (\mathcal{V}_B - 1)d/2} \\ &\times \mathcal{F}_{\mathcal{G}}\{(\mathbf{R}_i - \mathbf{R}_1) S^{-1/2}, i = 2, \dots, \mathcal{V}_B\} \end{aligned} \quad (6.37)$$

in such a way that the total partition function (6.36) is recovered as the $\mathcal{V}_B - 1$ d -dimensional space integral (2.2), (2.3) (the origin $i = 1$ of the network being always fixed)

$$\mathcal{Z}^B(\mathcal{G}) = \int \prod_{i=2}^{\mathcal{V}_B} d^d R_i \mathcal{Z}^B\{\mathcal{G}, \mathbf{R}_i\} \tag{6.38}$$

Suppose now that we place the same (Brownian) network in the presence of the surface, with vertices $i = 1, \dots, \mathcal{V}_B$ at positions $\mathbf{R}_i = (\mathbf{r}_{\parallel i}, z_i)$. Then, the translational invariance being broken, the partition function (6.37) is generalized to

$$\mathcal{Z}^B\{\mathcal{G}, \mathbf{r}_{\parallel i}, z_i\} = (2\pi S)^{-\mathcal{L}d/2 - (\mathcal{V}_B - 1)d/2} \mathcal{F}_{\mathcal{G}}\{(\mathbf{r}_{\parallel i} - \mathbf{r}_{\parallel 1}) S^{-1/2}, z_i S^{-1/2}\}$$

What happens now if we let some of the vertices of \mathcal{G} approach the surface? Let us denote by $j = 1, \dots, \mathcal{V}_S$ these surface vertices chosen among the total \mathcal{V}_B bulk vertices in such a way that

$$\mathcal{V}_B = \mathcal{V} + \mathcal{V}_S$$

with \mathcal{V} vertices staying in the bulk as in (6.1). Let L_j be the number of legs of the surface vertex j , and let $z_j \rightarrow 0$ be its position near the plane wall. We have seen in Eq. (6.17) that *each* Brownian chain constituting an L -vertex contributes a vanishing factor z to the partition function in direct space. So we expect for a Brownian network approaching the wall an asymptotic behavior

$$\mathcal{Z}^B(\mathcal{G}, z_j, j = 1, \dots, \mathcal{V}_S) \sim \prod_{j=1}^{\mathcal{V}_S} (z_j)^{L_j} \tag{6.39}$$

since each L -chain vertex gives a factor z^L . In order to define a finite surface partition function as in (6.19), we need to differentiate with respect to each of the positions z_j , L_j times. More precisely [as we actually did in (6.25), (6.28)], we differentiate with respect to each of the L_j chains of a vertex j *separately*, as if they were independent, and then let them all go to $z_j = 0$ together. This requires a total number of differentiations

$$\mathcal{L}_S \equiv \# \text{ contact differentiations} = \sum_{j=1}^{\mathcal{V}_S} L_j = \sum_{L \geq 1} n_L^S L \tag{6.40}$$

where n_L^S is the number of L -vertices onto the surface. \mathcal{L}_S is thus the total number of extremities of polymer lines upon contact to the surface.

We now define the surface partition function as

$$\begin{aligned} \partial_{\perp} \mathcal{Z}^{S,B}(\mathcal{G}, \mathbf{r}_{\parallel i}, z_i, i = 1, \dots, \mathcal{V}; \mathbf{r}_{\parallel j}, j = 1, \dots, \mathcal{V}_S) \\ \equiv \prod_{j=1}^{\mathcal{V}_S} \left(\frac{\partial}{\partial z_j} \right)^{L_j} \Big|_{z_j=0} \mathcal{Z}^B(\mathcal{G}, \mathbf{r}_{\parallel i}, z_i; \mathbf{r}_{\parallel j}, z_j) \end{aligned} \tag{6.41}$$

Next we integrate over all the d -dimensional positions of the \mathcal{V} bulk vertices of type $L \geq 1$ [Eq. (6.1)] and over the $(d-1)$ -dimensional positions of $\mathcal{V}_s - 1$ surface vertices:

$$\mathcal{Z}^{\text{S,B}}(\mathcal{G}) = \int \prod_{i=1}^{\mathcal{V}} d^{d-1} r_{\parallel i} \int_0^\infty \prod_{i=1}^{\mathcal{V}} dz_i \int \prod_{j=1}^{\mathcal{V}_s-1} d^{d-1} r_{\parallel j} \partial_\perp \mathcal{Z}^{\text{S,B}}(\mathcal{G}, \dots) \quad (6.42)$$

So, with respect to the bulk partition function (6.36), (6.38) we have a *reduction* of the number of space integration variables which is one per surface vertex, and thus in total $\mathcal{V}_s - 1$, i.e., *the total number of vertices on the surface minus one*, which is always kept fixed for eliminating translational invariance. Also taking into account the normal derivatives (6.40), (6.41), each reducing the length dimension by 1, i.e., giving a factor $S^{-1/2}$, we find the surface Brownian scaling behavior

$$\mathcal{Z}^{\text{S,B}}(\mathcal{G}) \sim S^{\gamma^{\text{S,B}}(\mathcal{G})-1} \quad (6.43)$$

$$\gamma^{\text{S,B}}(\mathcal{G}) - 1 = -\mathcal{L}d/2 - \sum_{L \geq 1} n_L^{\text{S}} L/2 - (\mathcal{V}_s - 1)/2$$

which replaces the bulk behavior. \mathcal{L} is still the total number of independent constitutive loops (6.1).

We are now in a position to study the direct renormalization of surface operators in the presence of excluded volume.

6.3. Direct Renormalization at a Surface

6.3.1. Dimensional Analysis. We consider now the full problem of polymer networks in the presence of an impenetrable wall, and with *excluded volume* as in (2.1). More specifically, here we consider the ordinary surface transition, i.e., *Dirichlet* boundary conditions (6.14). The Brownian analysis above, which led to Eq. (6.43), was a purely *dimensional* one. So we can assert directly that in the presence of excluded volume b , with a dimensionless interaction parameter z [Eq. (2.5)], the surface partition function will necessarily be of the form

$$\mathcal{Z}^{\text{S}}(\mathcal{G}) = (2\pi S)^{-d\mathcal{L}/2 - \mathfrak{g}_s/2 - (\mathcal{V}_s - 1)/2} Z^{\text{S}}(\mathcal{G}, z, d) \quad (6.44)$$

where Z^{S} is a dimensionless surface partition function depending on z only (and d). This holds true in *dimensional regularization*, and the arguments for obtaining (6.44) are similar to those of Sections 2.2 and 2.3 which led from the Brownian result $\mathcal{Z}^{\text{B}}(\mathcal{G})$ to the general formula (2.14) in the presence of excluded volume.

6.3.2. Partition Function of Surface Stars. We proceed now as in Section 3.2. First make a *hyperscaling* factorization as in Eq. (3.13) and rewrite (6.44) as

$$\mathcal{Z}^S(\mathcal{G}) = (2\pi R^2/d)^{-(d\mathcal{L} + \mathcal{V}_S + \nu_S - 1)/2} Z_v^S(\mathcal{G}, z, d) \tag{6.45}$$

where the true swollen size R^2/d [Eq. (3.3)] of a chain with excluded volume replaces the bare Brownian value S [Eq. (3.2)]. The new vertex part Z_v^S is of course related trivially to Z^S by

$$Z_v^S(\mathcal{G}) = [\mathcal{X}_0(z, d)]^{(d\mathcal{L} + \mathcal{V}_S + \nu_S - 1)/2} Z^S(\mathcal{G}) \tag{6.46}$$

This is not enough to renormalize the network partition function (6.44), and we have to factorize out, as in Eq. (3.17), the contributions of all the $n_L L$ -star vertices in the bulk for any $L \geq 1$, and also the new divergences brought in by the $n_L^S L$ -star vertices onto the surface, and this for any $L \geq 1$. So we first consider a simple L -star grafted onto the surface by its *core* (Fig. 15). It has an L -leg vertex grafted onto the surface, and L one-leg vertices floating in the bulk, and no loops; hence,

$$\mathcal{L} = 0, \quad \mathcal{V}_S = L, \quad \nu_S = 1 \tag{6.47}$$

Hence, according to (6.44), (6.45), its surface partition function scales like

$$\mathcal{Z}^S(\mathcal{S}_L) = (2\pi S)^{-L/2} Z^S(\mathcal{S}_L, z, d) \tag{6.48}$$

$$= (2\pi R^2/d)^{-L/2} Z_v^S(\mathcal{S}_L, z, d) \tag{6.49}$$

We choose now to rewrite Z_v^S by factorizing out the contributions of the L single extremities floating in the bulk, as in Eq. (3.15):

$$Z_v^S(\mathcal{S}_L, z, d) \equiv \hat{Z}_L^S(z, d) [\mathcal{Z}(\mathcal{S}_1)]^{L/2} \tag{6.50}$$

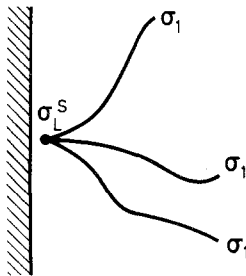


Fig. 15. Depiction of the scaling law (6.58), $\gamma_L^S - 1 = \sigma_L^S + L\sigma_1 - \nu L$, giving the L -arm star surface configuration exponent. σ_L^S is the irreducible contribution of the L -leg surface vertex.

Hence, the new irreducible surface partition function \hat{Z}_L^S of an L -vertex is explicitly defined by

$$\hat{Z}_L^S(z, d) = (2\pi R^2/d)^{L/2} \mathcal{Z}^S(\mathcal{G}_L) [\mathcal{Z}(\mathcal{G}_1)]^{-L/2} \tag{6.51}$$

or, in terms of the dimensionless partition functions (6.46) and (3.16),

$$\hat{Z}_L^S(z, d) = [\mathcal{X}_0(z, d)]^{L/2} Z^S(\mathcal{G}_L, z, d) [\hat{Z}_1(z, d)]^{-L} \tag{6.52}$$

So, we see the scaling idea clearly involved in (6.51); *one first factorizes out the hyperscaling dependence (6.45) of the surface star $\mathcal{Z}^S(\mathcal{G}_L)$, and then the contributions of the L free bulk ends, so one finally reaches the irreducible surface contribution \hat{Z}_L^S of the L -vertex.*

All these manipulations were only formal up to now, and are just a matter of definitions, based on intuition. Now the heart of the matter will come from the next general renormalization statement, valid for any network grafted onto the surface.

6.3.3. Renormalization by Surface Stars. Let us consider now the most general network \mathcal{G} with n_L L -leg vertices, $L \geq 1$, floating in the bulk, and n_L^S similar ones in mobile contact along the surface. We can always factorize it as [see (6.45) and (3.17)]

$$\begin{aligned} \mathcal{Z}^S(\mathcal{G}) &= [2\pi S \mathcal{X}_0(z, d)]^{-(d\mathcal{L} + \mathcal{V}_S + \mathcal{V}_S - 1)/2} \prod_{L \geq 1} [\hat{Z}_L(z, d)]^{n_L} \\ &\times \prod_{L \geq 1} [\hat{Z}_L^S(z, d)]^{n_L^S} \mathcal{A}^S(\mathcal{G}, z, d) \end{aligned} \tag{6.53}$$

where we have singled out the hyperscaling law (6.45), the irreducible contributions (3.15) of the L -leg vertices in the bulk, and those [(6.51), (6.52)] of the surface L -leg vertices of the network. This is just a definition of the dimensionless function \mathcal{A}^S . Note also that we can rewrite (6.53) more explicitly [use (3.15), (6.51), and (6.40)] in terms of the full star partition functions

$$\begin{aligned} \mathcal{Z}^S(\mathcal{G}) &= (2\pi R^2/d)^{-(d\mathcal{L} + \mathcal{V}_S - 1)/2} \prod_{L \geq 1} [\mathcal{Z}(\mathcal{G}_L) \mathcal{Z}^{-L/2}(\mathcal{G}_1)]^{n_L} \\ &\times \prod_{L \geq 1} [\mathcal{Z}^S(\mathcal{G}_L) \mathcal{Z}^{-L/2}(\mathcal{G}_1)]^{n_L^S} \mathcal{A}^S(\mathcal{G}) \end{aligned} \tag{6.54}$$

[Note that the factor $(2\pi R^2/d)^{-\mathcal{V}_S/2}$ cancels out in this latter formulation, but here $\mathcal{Z}^S(\mathcal{G}_L)$, (6.49), is dimensional.]

The important statement, as in Section 3.2, is that the factorization (6.53), (6.54) renormalizes correctly any surface network \mathcal{G} , namely now

$\mathcal{A}^S(\mathcal{G}, z, d)$ reaches a finite fixed point limit for very long chains when $z \rightarrow \infty$,

$$\mathcal{A}^S(\mathcal{G}, z, d) \xrightarrow{z \rightarrow \infty} \mathcal{A}^{*S}(\mathcal{G}, \infty, d) < \infty \tag{6.55}$$

In terms of the ε expansion, this means that $\mathcal{A}^S(\mathcal{G}, z, d)$, once expressed in terms of the renormalized g of (3.10) or z_R of (3.12) instead of z , has a double Taylor series expansion in powers of g or z_R , and ε , regular when $\varepsilon \rightarrow 0$, and this to all orders in g or z_R :

$$\mathcal{A}^S(\mathcal{G}, z, d) = \mathcal{A}^S[\mathcal{G}, g, d] = \mathcal{A}^S[\mathcal{G}, z_R, d]$$

singular in $1/\varepsilon$ regular regular

with a fixed point limit

$$\mathcal{A}^S(z \rightarrow \infty, d = 4 - \varepsilon) = \mathcal{A}^S[g^*, \varepsilon] = \mathcal{A}^S[z_R^*, \varepsilon]$$

Again, this statement is not trivial, and would require a whole proof to all orders, going deep into the renormalization theory. However, one can be confident that it holds actually true to all orders. For a single chain grafted onto the surface by one or two extremities, it can be shown easily to be completely equivalent to the usual field-theoretic approach,^(56,59) following, e.g., the method of ref. 35. A second argument is that the scaling laws derived from it in two dimensions, where the basic surface scaling dimensions are known exactly,⁽²²⁾ have been excellently checked numerically for various configurations of networks grafted onto a surface, or in a wedge, by Colby *et al.*⁽⁶⁶⁾

Let us now derive from (6.53), (6.54) the surface $\gamma_{\mathcal{G}}^S$ exponents of the network, and in particular show how scaling relations like (6.31) generalizing Barber's, (6.24), are readily obtained.

6.3.4. Basic Surface Exponents. Let us consider first a star polymer grafted onto a surface as in Fig. 15 by the L -leg core. We expect its partition function to scale like [see (6.2)]

$$\mathcal{Z}^S(\mathcal{G}_L) \sim \mu^{LS} S^{\gamma_L^S - 1} \tag{6.56}$$

where in dimensional regularization $\mu \equiv 1$, and where γ_L^S is the star surface exponent. If we consider instead the proper irreducible vertex function as defined in (6.51), it scales for z large like

$$\hat{\mathcal{Z}}_L^S(z, d) \xrightarrow{z \rightarrow \infty} \hat{A}_L^S(\varepsilon) z^{(2/\varepsilon)\sigma_L^S} \sim S^{\sigma_L^S} \tag{6.57}$$

where $\hat{A}_L^S(\varepsilon)$ is a calculable amplitude, and where the σ_L^S are new surface critical exponents, characteristic of the ordinary surface transition,

depending only on the number L of lines approaching the surface, and generalizing the similar bulk exponents σ_L , (3.30). We can immediately relate this surface dimension σ_L^S to the full γ_L^S exponent (6.56) of the surface star through (6.51), which readily implies

$$\gamma_L^S - 1 = \sigma_L^S + L\sigma_1 - \nu L$$

or

$$\sigma_L^S = \nu L + \gamma_L^S - 1 - \frac{1}{2}L(\gamma - 1) \tag{6.58}$$

So we see that σ_L^S is essentially related to γ_L^S , and the use of σ_L^S is a matter of combinatorial convenience, σ_L^S being related to the surface critical phenomena only, and being thought to embody the irreducible surface divergence associated with an L -leg polymer vertex. Note that for $L = 1$, we have

$$\gamma_1^S \equiv \gamma_1$$

where γ_1 is the usual magnetic surface exponent (here for the $n = 0$ field theory) and thus (6.58) reads

$$\sigma_1^S = \nu/2 + \gamma_1 - \gamma/2 - 1/2 \tag{6.59}$$

Now, we insert the scaling behaviors (3.5), (3.30), and (6.57) into (6.53) and find

$$\begin{aligned} \mathcal{Z}^S(\mathcal{G}) &\xrightarrow{z \rightarrow \infty} [2\pi S A_0(\varepsilon) z^{(2\nu-1)2/\varepsilon}]^{-(d\mathcal{L} + \mathfrak{L}_S + \mathcal{V}_S - 1)/2} \\ &\times \prod_{L \geq 1} [\hat{A}_L(\varepsilon) z^{\sigma_L(2/\varepsilon)}]^{n_L} \\ &\times \prod_{L \geq 1} [\hat{A}_L^S(\varepsilon) z^{\sigma_L^S(2/\varepsilon)}]^{n_L^S} \mathcal{A}^{*S}(\mathcal{G}) \end{aligned} \tag{6.60}$$

In terms of S , this gives simply

$$\mathcal{Z}^S(\mathcal{G}) \sim S^{-\nu(d\mathcal{L} + \mathfrak{L}_S + \mathcal{V}_S - 1) + \sum_{L \geq 1} (n_L \sigma_L + n_L^S \sigma_L^S)}$$

so that

$$\gamma_{\mathcal{G}}^S - 1 = -\nu(d\mathcal{L} + \mathfrak{L}_S + \mathcal{V}_S - 1) + \sum_{L \geq 1} (n_L \sigma_L + n_L^S \sigma_L^S) \tag{6.61}$$

If we use instead of (6.53) the equivalent scaling equation (6.54) and use

the star bulk and surface scaling behaviors (3.29) and (6.56), we readily find

$$\begin{aligned} \gamma_{\mathcal{S}}^{\mathcal{S}} - 1 &= -v(d\mathcal{L} + \mathcal{V}_{\mathcal{S}} - 1) \\ &+ \sum_{L \geq 1} \left\{ n_L \left[\gamma_L - 1 - \frac{L}{2}(\gamma - 1) \right] + n_L^{\mathcal{S}} \left[\gamma_L^{\mathcal{S}} - 1 - \frac{L}{2}(\gamma - 1) \right] \right\} \end{aligned} \tag{6.62}$$

which of course is completely equivalent to (6.61), due to (3.31) and (6.58).

We can finally rewrite $\gamma_{\mathcal{S}}^{\mathcal{S}}$ in (6.61) in terms of the bulk and surface vertex contributions by replacing \mathcal{L} , $\mathcal{Q}_{\mathcal{S}}$, and $\mathcal{V}_{\mathcal{S}}$ by their topological expressions (6.1) and (6.40). We find

$$\gamma_{\mathcal{S}}^{\mathcal{S}} - 1 = -vd + v + \sum_{L \geq 1} (n_L \Delta_L + n_L^{\mathcal{S}} \Delta_L^{\mathcal{S}}) \tag{6.63}$$

where, as in Eq. (3.34bis,ter)

$$\begin{aligned} \Delta_L &= \sigma_L - v \frac{d}{2} (L - 2) \\ &= \gamma_L - 1 - \frac{L}{2}(\gamma - 1) - v \frac{d}{2} (L - 2) \end{aligned}$$

and for the new surface contributions

$$\begin{aligned} \Delta_L^{\mathcal{S}} &= \sigma_L^{\mathcal{S}} - v \frac{d}{2} (L - 2) - v(L + 1) \\ &= \gamma_L^{\mathcal{S}} - 1 - \frac{L}{2}(\gamma - 1) - v \frac{d}{2} (L - 2) - v \end{aligned} \tag{6.64}$$

6.3.5. Various Hyperscaling Relations. We see that according to the general vertex structure⁽²²⁾ of the surface exponent $\gamma_{\mathcal{S}}^{\mathcal{S}}$ in (6.61), (6.62), we can express it for any network either in terms of the irreducible vertex contributions (6.61), or in terms of the more standard exponents of the stars γ_L in the bulk, of the surface stars $\gamma_L^{\mathcal{S}}$, and of v . In particular, (6.62) shows that any $\gamma_{\mathcal{S}}^{\mathcal{S}}$ can be expressed in terms of the basic exponents $\{v, \gamma_L, \gamma_L^{\mathcal{S}}, L \geq 1\}$. (Recall that for $L = 1$, $\gamma_{L=1} \equiv \gamma$ in the bulk, and $\gamma_{L=1}^{\mathcal{S}} \equiv \gamma_1$ in standard notations.) Of course, *a vertex decomposition law like (6.61), (6.62) generates infinities of (hyper)scaling relations between exponents.* This yields infinities of generalizations of Barber’s scaling law, the latter appearing as a consequence of the vertex formalism. Indeed, for a single chain attached to the surface by its *two* extremities, we have the

topological characteristics $n_L = 0, \forall L \geq 1$, and $n_1^S = 2, n_{L \neq 1}^S = 0, \mathcal{L} = 0, \mathcal{V}_S = 2$. Hence Eq. (6.62) gives γ_{11} as

$$\gamma_{11} - 1 = -v + 2[\gamma_1 - 1 - \frac{1}{2}(\gamma - 1)]$$

which is nothing but $2\gamma_1 - \gamma_{11} = \gamma + v$, as expected. We can easily generalize this, for instance, to the case of an L -line *watermelon* grafted onto a surface by its two extremities (Fig. 12). Its topological characteristics are $n_L = 0, \forall L \geq 1, n_L^S = 2, n_{L \neq L'}^S = 0, \mathcal{L} = L - 1, \mathcal{V}_S = 2$. Hence, Eq. (6.62) gives

$$\gamma_{\mathcal{W}_L}^S - 1 = -v[d(L - 1) + 1] + 2\left[\gamma_L^S - 1 - \frac{L}{2}(\gamma - 1)\right]$$

which can be rewritten as (with $\gamma_L^S \equiv \gamma_{\mathcal{S}_L}^S$)

$$2\gamma_{\mathcal{S}_L}^S - \gamma_{\mathcal{W}_L}^S = L\gamma + v + (L - 1)(vd - 1) \tag{6.65}$$

and is the full generalization of Barber’s, mentioned in Section 6.2 (Fig. 14).

Two tasks remain to be done now. First, relate these surface exponents σ_L^S which “act” in the “size space” S of the chains to the surface exponents x_L^S (6.4) in direct space; and, second, calculate them by renormalization theory.

6.3.6. Relation to the Surface Exponents x_L^S . We can check that the general scaling law (6.61) obtained for $\gamma_{\mathcal{S}}^S$ from direct renormalization theory is identical to the result⁽²²⁾ announced in (6.6) in terms of the exponents x_L^S , (6.4). Identifying the general structure (6.4) and (6.61) [see also the topological relations (6.1)] leads immediately to the identity

$$\sigma_L^S = -vx_L^S + (vd - 1)L/2 + vL \tag{6.66}$$

We can check this identity by recalling that the x_L^S exponents (6.4) measure the algebraic critical decay along the surface of the correlations of the extremities of an infinite watermelon network of fluctuating lengths (Fig. 12)

$$G_L^S(\mathbf{X} - \mathbf{Y}, T_c) = |\mathbf{X} - \mathbf{Y}|^{-2x_L^S}$$

where G_L^S is defined [Eq. (6.32)] by Laplace transform with respect to the total fluctuating length, and is a correlator in the associated $O(n)$ field theory ($n \rightarrow 0$). Using the same equations as in Sections 2.6 and 3.6, it is easy to invert this Laplace transform slightly off criticality [see (2.43)]

$$G_L^S(\mathbf{X} - \mathbf{Y}, T) = |\mathbf{X} - \mathbf{Y}|^{-2x_L^S} F_L^S(|\mathbf{X} - \mathbf{Y}| |T - T_c|^\nu)$$

and calculate the polydisperse number of configurations [Eq. (2.38)]

$$\mathcal{Z}_L^{S, \text{poly}}(S) = \frac{1}{2\pi i} \int_{-i\infty + \sigma}^{+i\infty + \sigma} dT e^{TS} \int d^{d-1} Y G_L^S(\mathbf{X} - \mathbf{Y}, T)$$

where now Y is integrated along the surface only. One finds therefore

$$\gamma_{\mathcal{W}_L}^{S, \text{poly}} = \nu(d - 1 - 2x_L^S)$$

and for the monodisperse case [Eq. (2.39)]

$$\gamma_{\mathcal{W}_L}^S - 1 = \nu(d - 1 - 2x_L^S) - L$$

Identifying this value with the result of Eq. (6.61) for \mathcal{W}_L , we just recover Eq. (6.66). QED. Note that it is also convenient to use the *anomalous* part $x_L^{\prime S}$ (6.8) of the correlation exponent x_L^S generated by self-avoidance. We get an equivalent surface identity from (6.66), which we collect with its bulk analogue (3.53):

$$\begin{aligned} \sigma_L^S &= -\nu x_L^{\prime S} + (\nu - 1/2)L, & \text{surface} \\ \sigma_L &= -\nu x_L' + (\nu - 1/2)L, & \text{bulk} \end{aligned} \tag{6.67}$$

These basic identities give the relation between the “size space” vertex exponents σ_L, σ_L^S appearing in the general formulas (3.34), (6.61) for $\gamma_{\mathcal{G}}, \gamma_{\mathcal{G}}^S$ and the *pure anomalous* scaling dimensions $x_L', x_L^{\prime S}$ of the bulk and surface vertex operators such that

$$\begin{aligned} \langle \phi_L(\mathbf{X}) \phi_L(\mathbf{Y}) \rangle_{T_c} &= |\mathbf{X} - \mathbf{Y}|^{-(d-2)L - 2x_L'} \\ \langle \phi_L^S(\mathbf{X}) \phi_L^S(\mathbf{Y}) \rangle_{T_c} &= |\mathbf{X} - \mathbf{Y}|^{-dL - 2x_L^{\prime S}} \end{aligned} \tag{6.68}$$

Recall that $-(d-2)L$ and $-dL$ are the Brownian exponents. Hence σ_L, σ_L^S , as well as $x_L', x_L^{\prime S}$ are *identically zero above four dimensions, and start at order ϵ , in $d = 4 - \epsilon$* . They are generated by self-avoidance.

6.4. Calculation of the Exponents

In direct renormalization the easiest way is to calculate the surface configuration exponents of the L -stars $\gamma_{\mathcal{S}_L}^S$, which will lead to the σ_L^S , in the same way as we calculated in Sections 3 and 5 the bulk exponents σ_L . We perform here this calculation to first order in ϵ .

6.4.1. Perturbation Expansion. The rules for calculating diagrams in the presence of a surface are the same as in Section 2.4 except for two important changes.

1. In the bulk, we took as a free propagator between two successive interaction points of the same chain in a diagram the Gaussian form (2.20) (in Fourier space) or in direct space

$$\langle \delta^d(\mathbf{r}_a(s) - \mathbf{r}_a(s') - \mathbf{r}) \rangle_0 = (2\pi |s - s'|)^{-d/2} \exp(-r^2/2|s - s'|) \quad (6.69)$$

This bulk Gaussian probability distribution has to be replaced in the presence of a surface by the Brownian distribution G_0^S of (6.12). For the present ordinary surface transition or *Dirichlet* boundary conditions we replace in the calculation of the diagrams (6.69) by (6.15):

$$\begin{aligned} G_0^S(\mathbf{r}, z, \mathbf{r}', z'; s - s') &= (2\pi |s - s'|)^{-d/2} \exp \left[-\frac{1}{2|s - s'|} (\mathbf{r} - \mathbf{r}')^2 \right] \\ &\times \left\{ \exp \left[-\frac{1}{2|s - s'|} (z - z')^2 \right] - \exp \left[-\frac{1}{2|s - s'|} (z + z')^2 \right] \right\} \end{aligned} \quad (6.70)$$

In general the best method would be to Fourier transform the part parallel to the surface (depending on $\mathbf{r} - \mathbf{r}'$ only) and use in the general calculation of diagrams a mixed Fourier–direct space representation, since the z components break the translational invariance. However, to first order, all calculations can be performed in direct space.

2. As explained in detail above, for Dirichlet boundary conditions, we have to differentiate the Brownian Green functions upon contact at the surface, in order to define finite partition functions [Eqs. (6.18), (6.19), (6.25), (6.28)]. So when we calculate, for example, the diagrams contributing to an L -star partition function (Fig. 16), we take as a *first*

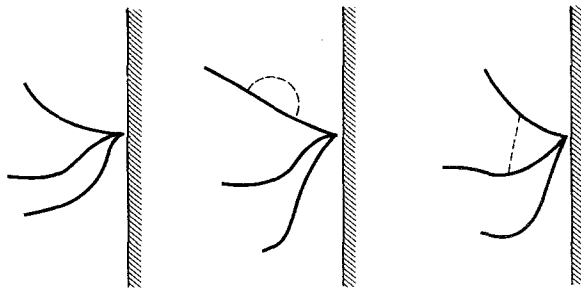


Fig. 16. One-loop diagrams contributing to the star surface partition function $\mathcal{Z}^S(\mathcal{L}_L)$, and parametrizing \mathfrak{J} and \mathcal{J} .

propagator for any segment of length s joining the *core* on the surface to a first interaction point the propagator (6.18):

$$\left. \frac{\partial}{\partial z'} \right|_{z'=0} G_0^S(0, 0, \mathbf{r}, \mathbf{z}; s) = \frac{2z}{s} (2\pi s)^{-d/2} \exp \left[-\frac{1}{2s} (\mathbf{r}^2 + z^2) \right] \quad (6.71)$$

6.4.2. Surface Exponents at $O(\epsilon)$. According to the theory described before, we only need to know the surface star partition $\mathcal{L}_{\mathcal{S}_L}^S$ characterizing an L -leg star attached by its core to the surface. The first-order diagrams contributing to it are given in Fig. 16, up to one loop. Let us denote by \mathfrak{J}_L and \mathcal{J}_L their contributions. When we take into account their multiplicity weights, we find for $\mathcal{L}_{\mathcal{S}_L}^S$

$$\mathcal{L}^S(\mathcal{S}_L) = \mathcal{L}^{S,B}(\mathcal{S}_L) + L\mathfrak{J}_L + \frac{1}{2}L(L-1)\mathcal{J}_L \quad (6.72)$$

The zeroth-order contribution is the Brownian value (6.28) for $b=0$

$$\mathcal{L}^{S,B}(\mathcal{S}_L) = [2(2\pi S)^{-1/2}]^L \quad (6.73)$$

According to the rules above, \mathfrak{J}_L reads explicitly

$$\mathfrak{J}_L = \mathfrak{J} \mathcal{L}_{\mathcal{S}_{L-1}}^{S,B} = \mathfrak{J} [2(2\pi S)^{-1/2}]^{L-1} \quad (6.74)$$

where \mathfrak{J} is the contribution for one grafted chain only (Fig. 17a). In the same way, \mathcal{J}_L factorizes into

$$\mathcal{J}_L = \mathcal{J} \mathcal{L}_{\mathcal{S}_{L-2}}^{S,B} = \mathcal{J} [2(2\pi S)^{-1/2}]^{L-2} \quad (6.75)$$

where \mathcal{J} is the contribution of the two-leg star grafted by its core (Fig. 17b).

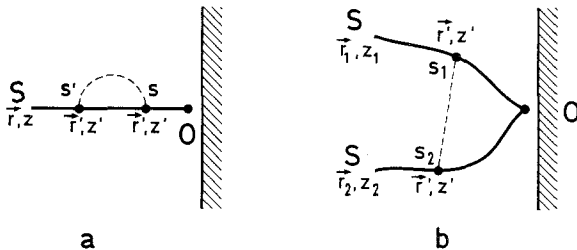


Fig. 17. Parametrization of interaction integrals \mathfrak{J} and \mathcal{J} occurring in (6.74), (6.75).

According to the diagrammatic rules above, the integral expression of \mathfrak{J} is formally (Fig. 17a)

$$\begin{aligned} \mathfrak{J} &= -b \int_0^S ds \int_s^S ds' \int d^{d-1}r \int_0^\infty dz \int d^{d-1}r' \int_0^\infty dz' \\ &\quad \times \frac{\partial}{\partial z} G_0^S(0, z=0, r', z', s) \\ &\quad \times G_0^S(r', z', r', z'; s' - s) G_0^S(r', z', \mathbf{r}, z; S - s') \end{aligned}$$

Using the expressions (6.70), (6.71), we find explicitly

$$\begin{aligned} \mathfrak{J} &= -b \int_0^S ds \int_s^S ds' \int d^{d-1}r' \int_0^\infty dz' \frac{2z'}{s} (2\pi s)^{-d/2} e^{-(r')^2/2s} e^{-(z')^2/2s} \\ &\quad \times (1 - e^{-4(z')^2/2(s'-s)}) [2\pi(s'-s)]^{-d/2} \\ &\quad \times \int d^{d-1}r \int_0^\infty dz G_0^S(r', z', \mathbf{r}, z; S - s') \end{aligned}$$

The last integration over the free extremity (\mathbf{r}, z) is performed with the help of the identity (6.16), while that on r' is trivial. So \mathfrak{J} reads finally

$$\begin{aligned} \mathfrak{J} &= -b \int_0^S ds \int_s^S ds' \int_0^\infty dz' \\ &\quad \times \frac{2z'}{s} (2\pi s)^{-1/2} e^{-(z')^2/2s} [1 - e^{-4(z')^2/2(s'-s)}] \\ &\quad \times [2\pi(s'-s)]^{-d/2} [2\pi(S-s')]^{-1/2} 2 \int_0^{z'} dz'' e^{-(z'')^2/2(S-s')} \end{aligned}$$

This integral, once dimensionally regularized, diverges when $d = 4 - \varepsilon$, $\varepsilon \rightarrow 0$. One can perform its asymptotic analysis and find

$$\mathfrak{J} = -2(2\pi S)^{-1/2} z [-2/\varepsilon + O(1)] \tag{6.76}$$

where the term $O(1)$ is calculable and useful for calculating some universal amplitudes.⁽⁷⁰⁾ Hence, from (6.74)

$$\mathfrak{J}_L = [2(2\pi S)^{-1/2}]^L z \frac{2}{\varepsilon} \tag{6.77}$$

The partition function \mathcal{J} for the grafted two-leg star (Fig. 17) is calculated in the same way. Its formal expression reads

$$\begin{aligned} \mathcal{J} &= -b \int_0^S ds_1 \int_0^S ds_2 \int d^{d-1}r' \int_0^\infty dz' \\ &\times \frac{\partial}{\partial z} G_0^S(0, 0, \mathbf{r}', z'; s_1) \frac{\partial}{\partial z} G_0^S(0, 0, \mathbf{r}', z'; s_2) \\ &\times \int d^{d-1}r_1 \int_0^\infty dz_1 G_0^S(\mathbf{r}', z', \mathbf{r}_1, z_1; S-s_1) G_0^S(\mathbf{r}', z', \mathbf{r}_2, z_2; S-s_2) \end{aligned}$$

Using the same trick as before [Eq. (6.16)] to integrate over $\mathbf{r}_1, z_1, \mathbf{r}_2, z_2$, and \mathbf{r}' , we find the simplified expression

$$\begin{aligned} \mathcal{J} &= (2\pi)^{-(d+3)/2} 2^4 \int_0^S ds_1 (S-s_1)^{-1/2} s_1^{-d/2-1} \\ &\times \int_0^S ds_2 (S-s_2)^{-1/2} s_2^{-d/2-1} \int_0^\infty dz' (z')^2 \\ &\times \int_0^{z'} d\zeta e^{-\zeta^2/2(S-s_1)} \int_0^{z'} d\zeta' e^{-(\zeta')^2/2(S-s_2)} \\ &\times (s_1^{-1} + s_2^{-1})^{-(d-1)/2} e^{-(z')^2 (s_1^{-1} + s_2^{-1})/2} \end{aligned}$$

The analysis of the divergences when $d = 4 - \varepsilon, \varepsilon \rightarrow 0$, leads, after some calculations, to

$$\mathcal{J} = -(2\pi S)^{-1} z \frac{2^3}{\varepsilon} \tag{6.78}$$

Inserting this into Eq. (6.75), we find

$$\mathcal{J}_L = -[2(2\pi S)^{-1/2}]^L z \frac{2}{\varepsilon} \tag{6.79}$$

Finally, we find for the star surface partition function (6.72) the simple expansion

$$\begin{aligned} \mathcal{Z}^S(\mathcal{L}_L) &= [2(2\pi S)^{-1/2}]^L \left[1 - zL(L-3) \frac{1}{\varepsilon} \right] \\ &\equiv [2(2\pi S)^{-1/2}]^L Z^S(\mathcal{L}_L, z, d) \end{aligned} \tag{6.80}$$

according to the definition (6.44). We only need now to renormalize this result. This is trivially done as in Section 3. We calculate the logarithmic derivative

$$\begin{aligned}
S \frac{\partial}{\partial S} \ln Z^S(\mathcal{G}_L, z, d) &= \frac{\varepsilon}{2} z \frac{\partial}{\partial z} \ln Z^S(\mathcal{G}_L) \\
&= -\frac{1}{2} z L(L-3) + \dots = -\frac{1}{2} g L(L-3) + \dots \quad (6.81)
\end{aligned}$$

which must be to all orders a regular (for $\varepsilon \rightarrow 0$) double series expansion of g and ε . Substituting the fixed point value (3.11), we find asymptotically for $d = 4 - \varepsilon$

$$S \frac{\partial}{\partial S} \ln Z^S(\mathcal{G}_L) \underset{S \rightarrow \infty}{\simeq} -\frac{1}{2} \frac{\varepsilon}{8} L(L-3) + O(\varepsilon^2)$$

and thus

$$\mathcal{Z}^S(\mathcal{G}_L) \sim [2(2\pi S)^{-1/2}]^L S^{-(\varepsilon/16)L(L-3)}$$

Therefore, the star surface exponent $\gamma_L^S \equiv \gamma_{\mathcal{G}_L}^S$ defined in (6.56) reads at this order

$$\gamma_{\mathcal{G}_L}^S - 1 = -\frac{L}{2} - \frac{\varepsilon}{16} L(L-3) + O(\varepsilon^2) \quad (6.82)$$

Note at this stage that the anomalous parts of the *surface exponents* $\gamma_{\mathcal{G}_L}^S$ and of the *bulk exponents* $\gamma_{\mathcal{G}_L}$ of (5.17) for star polymers are the same to order $O(\varepsilon)$. This does not hold true to the next orders. This means that the polar divergences of the surface diagrams of Fig. 16a are the same as those of the bulk diagrams of Fig. 7. So, in the vertex decomposition formula (6.62), the γ_L^S surface exponents could be replaced by the γ_L of stars in the bulk, up to terms of order $O(\varepsilon^2)$.

The vertex irreducible surface exponent σ_L^S is then calculated from (6.58) and (1.3) as

$$\sigma_L^S = -\frac{\varepsilon}{16} L(L-3) + O(\varepsilon^2) \quad (6.83)$$

and from this value, the x_L^S and $x_L'^S$ exponents are easily derived [Eqs. (6.67)]

$$\begin{aligned}
x_L^S &= d \frac{L}{2} + \frac{\varepsilon}{8} L(L-2) + O(\varepsilon^2) \\
x_L'^S &= \frac{\varepsilon}{8} L(L-2) + O(\varepsilon^2)
\end{aligned} \quad (6.84)$$

as announced in (6.9).

From these values, one has access to any surface critical exponent of any network \mathcal{G} . For instance, the global Δ_L , Δ_L^S exponents (6.64) read at this order

$$\begin{aligned} \Delta_L &= -\frac{d}{4}(L-2) + \frac{\varepsilon}{16}(4-L^2) + O(\varepsilon^2) \\ \Delta_L^S &= -\frac{d}{4}(L-2) - \frac{L+1}{2} + \frac{\varepsilon}{16}(3-L^2) + O(\varepsilon^2) \end{aligned} \tag{6.85}$$

6.4.3. Four Dimensions. The logarithmic terms correcting the Brownian behavior in $d=4$ (4D) of $\mathcal{Z}^S(\mathcal{S}_L)$ [Eq. (6.80)] can be calculated as in Section 3, by replacing g in (6.81) by its asymptotic form (3.45), $g = 1/(4 \ln S) + \dots$. Integrating back gives

$$\mathcal{Z}^S(\mathcal{S}_L) \underset{S \rightarrow \infty}{\underset{d=4}{\sim}} [2(2\pi S)^{-1/2}]^L (\ln S)^{-L(L-3)/8} \tag{6.86}$$

Again, this is similar to the exact $d=4$ asymptotic behavior of the star partition function in the bulk (since, as is well known, the dominant $d=4$ log terms are given by one-loop divergences).

Indeed, the general formula (3.47) gave for bulk star polymers in 4D

$$\mathcal{Z}(\mathcal{S}_L) \underset{S \rightarrow \infty}{\underset{d=4}{\sim}} (\ln S)^{-L(L-3)/8} \tag{6.87}$$

So the only change in the surface behavior is the occurrence of an $S^{-L/2}$ factor, due to the Dirichlet boundary conditions, which required L contact differentiations for the L arms of the star core (Section 6.2.2).

It is now interesting to find the exact scaling behavior in four dimensions for *any network grafted onto a Dirichlet surface*. For such a network \mathcal{G} , we use the basic multiplicative renormalization formula (6.54), in terms of the bulk and surface star partition functions, given in 4D in (6.86), (6.87). We need also the logarithmic behaviors in 4D⁽⁴³⁾ in the bulk [Eqs. (3.43)]

$$R^2 \sim S(\ln S)^{1/4}, \quad \mathcal{Z}(\mathcal{S}_1) \sim (\ln S)^{1/4} \tag{6.88}$$

and on the surface

$$\mathcal{Z}^S(\mathcal{S}_1) \sim 2(2\pi S)^{-1/2} (\ln S)^{1/4}$$

Now in (6.54), the amplitude $\mathcal{A}^S(\mathcal{G})$, as in (3.18), has been tailored in such a way that

$$\mathcal{A}^S = \mathcal{A}^{S, \text{Brownian}} + O(g) = \mathcal{A}^{S, B} + O(1/\ln S) \quad (d=4) \tag{6.89}$$

and thus \mathcal{A}^S also tends to a constant in 4D (its Brownian value). Collecting together (6.86)–(6.88) and using also the topological relations (6.1) yields after some calculations the new form of the multiplicative renormalization equation (6.54) in $d = 4$,

$$\mathcal{L}^S(\mathcal{G}) \underset{\substack{d=4 \\ S \rightarrow \infty}}{\sim} (2\pi S)^{-(4\mathcal{L} + \mathcal{Q}_S + \mathcal{V}_S - 1)/2} \times (\ln S)^{\sum_{L \geq 1} (n_L + n'_L)(4 - L^2)/8 - \mathcal{V}'_S/8 - 3/8} \tag{6.90}$$

where \mathcal{L} , $\mathcal{Q}_S = \sum_{L \geq 1} n'_L S^L$, $\mathcal{V}_S = \sum_{L \geq 1} n_L S^L$, are, respectively, the numbers of independent loops of \mathcal{G} , of chain ends grafted onto the surface, and of vertices grafted onto the surface. Of course, in the above formula, we recognize in the first factor the Brownian value (6.43) at $d = 4$.

6.5. Corollaries

From the above results we can derive some other interesting quantities.

6.5.1. A Remarkable Identity. Consider the two-leg surface vertices. From (6.84), we observe that the *anomalous* part of x_2^S , namely $x_2^{\prime S}$ vanishes to order ε :

$$x_2^{\prime S} = 0 \tag{6.91}$$

The same is true for the exact two-dimensional formula (6.10) derived from conformal invariance⁽²²⁾

$$x_L^{\prime S} = \frac{3}{8} L(L - 2), \quad x_2^{\prime S} \equiv 0 \quad (d = 2) \tag{6.91bis}$$

Actually, this identity (6.91) holds true to *all orders* in ε , and is valid in *any dimension*. It is indeed the translation in polymer physics, namely in the $O(n = 0)$ field theory, of a result of Diehl *et al.*⁽⁵⁷⁾ for the ordinary surface transition.

They showed that the *anomalous dimension* of the normal derivative of the field at the Dirichlet boundary $\partial_\perp \phi = \partial_z \phi|_{z=0}$ *vanishes to all orders* in ε . Consequently, the crossover exponent Φ^0 governing the effect of a surface energy term $c \int_{\text{surface}} \phi^2$ was shown to be $\Phi^0 = -\nu$ to all orders. Here, in polymer theory, $x_2^{\prime S}$ is just the anomalous dimension of the two-leg vertex in contact with the surface and corresponds in surface magnetism precisely to the operator $(\partial_\perp \phi)^2$. So (6.91) is just the same vanishing theorem, valid to all orders in ε , also exact in *two dimensions*, and in fact in *any dimension* d .

The related exponents $x_2^S, \sigma_2^S, \Delta_2^S$ [Eqs. (6.8), (6.68), (6.64)] are thus

$$\begin{aligned} x_2^S &= d \equiv x_2^{S, \text{Brownian}} \\ \sigma_2^S &= 2v - 1 \quad (\text{ordinary surface transition}) \\ \Delta_2^S &= -v - 1 \end{aligned} \tag{6.92}$$

We can derive from this the interesting values of the γ exponents of some networks near a surface when they involve only two-leg vertices on the surface (Fig. 18). We have the following for the networks of Fig. 18.

Single Loop Attached to the Surface. $n_L = 0, \forall L \geq 1, n_2^S = 1, n_{L \neq 2}^S = 0$. Using, e.g., Eq. (6.63), we find from (6.92)

$$\gamma_{\text{loop}}^S = -vd \tag{6.93}$$

Two-Leg Star Attached by Its Core. $n_1 = 2, n_2^S = 1$. Equation (6.58) for $L = 2$ and (6.92) give the identity

$$\gamma_{\mathcal{S}_2}^S = \gamma - 1 \tag{6.94}$$

Two-Leg Watermelon. $n_L = 0, \forall L \geq 1, n_2^S = 2$. Equation (6.63), e.g., gives

$$\gamma_{\mathcal{W}_2}^S = -vd - v - 1 \tag{6.95}$$

Linear Frieze. \mathcal{N} chains are tied as a collar onto the surface (Fig. 18). Hence the topological characteristics are $\mathcal{L} = 0, \mathcal{Q}_S = 2\mathcal{N}, \mathcal{V}_S = \mathcal{N} + 1, n_1^S = 2, n_2^S = \mathcal{N} - 1$, and thus

$$\gamma_{\text{frieze}} - 1 = -v3\mathcal{N} + 2\sigma_1^S + (\mathcal{N} - 1)\sigma_2^S$$

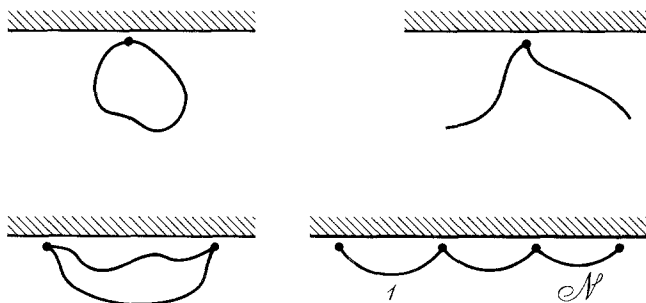


Fig. 18. Networks involving only two leg-stars grafted onto the surface (i.e., surface energy operators in field theory) and whose configuration exponent $\gamma_{\mathcal{S}}$ depends only on $\gamma, v, \text{ and } d$, and for the linear frieze also on γ_1 .

We recall that for a single chain attached by both extremities, in our formalism [see (6.64)]

$$\gamma_{11} - 1 \equiv \gamma_{\mathcal{N}_1}^{\mathcal{S}} - 1 = -3\nu + 2\sigma_1^{\mathcal{S}}$$

which is also the above equation for $\mathcal{N} = 1$. Hence, combining the latter two equations and using (6.92) gives

$$\begin{aligned} \gamma_{\text{frieze}}^{\mathcal{S}} &= \gamma_{11} + \nu - \mathcal{N}(\nu + 1) \\ &= 2\gamma_1 - \gamma - \mathcal{N}(\nu + 1) \end{aligned} \quad (6.96)$$

Circular Frieze. We can also consider a surface *circular* frieze obtained by gluing together the two free extremities of the above linear frieze. Then we have simply $n_2^{\mathcal{S}} = \mathcal{N}$, and

$$\begin{aligned} \gamma_{\text{frieze}}^{\mathcal{S}} &= 1 - \nu d + \nu + \mathcal{N} d_2^{\mathcal{S}} \\ &= 1 - \nu d + \nu - \mathcal{N}(\nu + 1) \end{aligned} \quad (6.97)$$

As it must, for $\mathcal{N} = 1$ we recover the loop result (6.93), and for $\mathcal{N} = 2$ the $L = 2$ watermelon result (6.95).

6.5.2. Proximal Exponents. From the knowledge of the $\gamma^{\mathcal{S}}$ exponents, one can derive that of the *proximal* exponents which govern the probability that a network approaches the surface. More precisely, one considers a particular L -leg vertex i belonging to a given network \mathcal{G} and defines the relative probability that it stays at a distance z from the surface,

$$\mathcal{P}_L(z) = \frac{\mathcal{Z}_{\mathcal{G}}(z)}{\mathcal{Z}_{\mathcal{G}}(\infty)} \quad (6.98)$$

where $\mathcal{Z}_{\mathcal{G}}(z)$ is the total partition function of the network, with the particular L -leg vertex i fixed at a distance z from the wall, $\mathcal{Z}_{\mathcal{G}}(z = \infty)$ corresponding to be same partition function at infinity, namely in the *bulk*. *A priori*, $\mathcal{Z}_{\mathcal{G}}(z)$ should depend also on the surroundings of the considered vertex in the network, i.e., on which L -leg vertex we choose to fix at z . However, when the vertex comes close to the surface, only the fact that it has L legs will matter, exactly as in the case of the local contacts inside a polymer network, the contact exponent $\theta_{\{L\}}$ [Eqs. (1.7), (5.32)] depended only on the vertices upon contact. This is the reason for the subscript L in \mathcal{P}_L .

Since $\mathcal{P}_L(z)$ is dimensionless by its very definition (6.98), it will depend, in the asymptotic limit of strong excluded volume, only on the ratio z/R , where the size [see (3.1)] $R \sim S^{\nu}$ of the swollen chain (e.g., in the

bulk) is the natural length scale of the associated critical system. Now, when the distance to the surface goes to zero, i.e., for $z/R \rightarrow 0$, \mathcal{P}_L is expected to scale like

$$\mathcal{P}_L(z) \underset{z \rightarrow 0}{\sim} (z/R)^{\zeta_L} \tag{6.99}$$

where ζ_L is a new critical exponent, the *proximal* or *depletion* exponent. It is a contact exponent to the surface and signals a *depletion* effect. As said above, this exponent depends only on L , the number of legs of the polymer vertex approaching the surface, and not at all on the network it is embedded in. To derive it, we simply use the same scaling argument⁽¹¹⁾ as for the contact exponents in Section 5.4. We fix z at a “lattice spacing” a , say, of the surface and let the size $R \sim S^v$ of the chain be very large. So from (6.99) we expect \mathcal{P}_L to scale like $\mathcal{P}_L \sim S^{-v\zeta_L}$, while from (6.98), we expect it to scale like

$$\mathcal{P}_L \sim S^{-v\zeta_L} \sim \mathcal{Z}_{\mathcal{G}'}^S / \mathcal{Z}_{\mathcal{G}} \sim S^{\gamma_{\mathcal{G}'}^S - \gamma_{\mathcal{G}}} \tag{6.100}$$

where $\mathcal{Z}_{\mathcal{G}'}^S$ is now the *surface* partition function of the new network \mathcal{G}' grafted onto the surface by the considered L -leg vertex and only by this one. Equation (6.100) implies therefore

$$\zeta_L = \frac{1}{v} (\gamma_{\mathcal{G}} - \gamma_{\mathcal{G}'}^S) \tag{6.101}$$

Suppose now that the numbers of L' -leg vertices of \mathcal{G} were $\{n_{L'}\}$ in the bulk and $\{n_{L'}^S = 0\}$ on the surface. Once \mathcal{G}' is grafted onto the surface by one L -leg vertex, one has the new primed values

$$\begin{aligned} n'_{L' \neq L} &= n_{L' \neq L}, & n'_L &= n_L - 1, & n'^S_{L' \neq L} &= 0, & n'^S_L &= 1 \\ \mathcal{V}' &= \mathcal{V} - 1, & \mathcal{V}'^S &= 1 \end{aligned} \tag{6.102}$$

In this problem, the most convenient formalism for expressing the bulk and surface exponents $\gamma_{\mathcal{G}}$ and $\gamma_{\mathcal{G}'}^S$ will be that in terms of the dimensions x_L [Eqs. (4.14bis), (4.14ter)] and of the $x_L x_L^S$ [Eq. (6.6)]. We therefore have in the bulk

$$\gamma_{\mathcal{G}} - 1 = v \left[d(\mathcal{V} - 1) - \sum_{L' \geq 1} n_{L'} x_{L'} \right] - \mathcal{N}$$

and once \mathcal{G}' is grafted onto the surface by one L -leg vertex

$$\gamma_{\mathcal{G}'}^S - 1 = v \left(d\mathcal{V}' - \sum_{L' \geq 1} n'_{L'} x_{L'} - x_L^S \right) - \mathcal{N}$$

Using (6.102) gives identically

$$\gamma_{\mathcal{G}} - \gamma_{\mathcal{G}'}^S \equiv v(x_L^S - x_L)$$

Hence, this value does not depend on \mathcal{G} , but only on L . We could have obtained it much more easily by only considering the *approach of the core of a simple L -leg star* to the surface. The corresponding proximal exponent reads, finally,

$$\zeta_L = x_L^S - x_L \quad (6.103)$$

which is a formula very similar to that for contact exponents (5.32). Both are actually simple operator product expansion formulas.

Recall that in $d = 4 - \varepsilon$ dimensions we found [Eqs. (5.20), (6.84), (6.9)]

$$x_L = \frac{L}{2}(d-2) + \frac{\varepsilon}{8}L(L-1) + O(\varepsilon^2)$$

$$x_L^S = \frac{L}{2}d + \frac{\varepsilon}{8}L(L-2) + O(\varepsilon^2)$$

Hence

$$\zeta_L = L(1 - \varepsilon/8) + O(\varepsilon^2) \quad (6.104)$$

Note that for $\varepsilon = 0$ we recover well the Brownian value $\zeta_L^B = L$. In this case, the probability $\mathcal{P}_L^B(z)$ of (6.98) for an L -leg Brownian star is just the partition function (6.17) raised to the L th power. Hence, near the surface

$$\mathcal{P}_L^B(z) \underset{z \rightarrow 0}{\approx} \left[2 \frac{z}{(2\pi S)^{1/2}} \right]^L$$

in agreement with (6.104).

In *two dimensions* we know the exact values of the scaling dimensions x_L, x_L^S :

$$x_L = (9L^2 - 4)/48, \quad x_L^S = L(3L + 2)/8$$

Hence the proximal exponents are exactly⁽²²⁾

$$\zeta_L = (3L + 2)^2/48 \quad (\text{dilute, 2D}) \quad (6.105)$$

The other critical phase in two dimensions, namely the dense polymer phase⁽¹³⁾—corresponds to the low-temperature phase of the associated

critical $O(n)$ model ($n \rightarrow 0$). The exact dense exponents are, in two dimensions,^(13,18,20)

$$x_L^D = (L^2 - 4)/16, \quad x_L^{D,S} = L(L - 2)/8$$

Hence the proximal exponents are in this case

$$\zeta_L = (L - 2)^2/16 \quad (\text{dense, 2D}) \tag{6.106}$$

For $L = 2$, we remark that $\zeta_2 = 0$. So there is no repulsion from the surface line acting onto the two-leg vertices, which are nothing but the *interior monomers* of the chains. This is entirely satisfactory since the polymer phase is dense and fills the box with a finite density⁽¹³⁾ and can even fill completely the lattice^(20,21) (Hamiltonian walks), with density one. So the dense polymers also fill the neighborhood of the surface and one expects a *no depletion* regime in the dense phase. The depletion which would occur in the dilute case due to the repelling of the boundary chains by the wall is counterbalanced here by the bulk dense monomers repelling the boundary ones.

7. POLYMER NETWORKS AT THE THETA POINT

7.1. Tricritical Model

Finally, one can generalize the above scaling theory to networks in a Θ -solvent. Since this paper is already long, I shall mainly concentrate on the results. The Θ point in a polymer is experimentally defined as the temperature where the second virial coefficient of (infinite) chains vanishes.^(71,24,2) From a theoretical point of view,^(2,24,25,72) it can be modeled by a continuum Edwards model⁽²⁵⁾ with three-body interactions

$$\begin{aligned} \mathcal{P}\{\mathbf{r}\} &= \exp(-\mathcal{A}\{\mathbf{r}\}) \\ \mathcal{A}\{\mathbf{r}\} &= \frac{1}{2} \int_0^S \left(\frac{d\mathbf{r}}{ds}\right)^2 ds + \frac{b}{2} \int_0^S ds \int_0^S ds' \delta^d[\mathbf{r}(s) - \mathbf{r}(s')] \\ &\quad + \frac{c}{3!} \int_0^S ds \int_0^S ds' \int_0^S ds'' \delta^d[\mathbf{r}(s) - \mathbf{r}(s')] \delta^d[\mathbf{r}(s') - \mathbf{r}(s'')] \end{aligned} \tag{7.1}$$

This weight for one chain is easily generalized to a set of \mathcal{N} chains as in (2.1). The dimensionless expansion parameters are,⁽²⁵⁾ for the two- and three-body forces, by trivial dimensional analysis of (7.1)

$$z = (2\pi)^{-d/2} bS^{2-d/2}, \quad y = (2\pi)^{-d} cS^{3-d} \tag{7.2}$$

So we see that for $d > 3$, the three-body forces are irrelevant in the long-chain limit. For $d \leq 3$ they are relevant. Thus $d = 3$ is upper tricritical dimension. One can then show,⁽⁷³⁾ that in *dimensional regularization*, the theta point corresponds exactly to the *vanishing* of the two-body parameter $b = 0$, i.e., $z = 0$. For simplicity, I shall work here exactly at this θ point, which is a *tricritical point*.^(24,72) By the same direct renormalization method, which I shall describe here, one could also calculate⁽²⁵⁾ tricritical deviations from the θ point, which are experimentally observable.⁽⁷⁴⁾ Before proceeding further into the theory of networks, let us recall briefly what is known about single polymer chains at the θ point.

Since $d = 3$ is the upper tricritical dimension, logarithmic corrections to the Brownian behavior appear in $d = 3$. The squared end-to-end distance and the single-chain partition function then read,^(25,75) exactly at the θ point, in *three* dimensions,

$$R^2 = 3SA_0(y) \left[1 - \frac{37}{3(11)^2 \ln(S/s_0)} \right] \quad (3D) \quad (7.3)$$

$$\mathcal{Z}^\theta(\mathcal{S}_1) = A_1(y) \left[1 - \frac{49}{4(11)^2 \ln(S/s_0)} \right] \quad (3D) \quad (7.3)$$

where A_0 , A_1 are regular functions of y , starting for y small as

$$A_0(y) = 1 + \frac{16}{33} \pi y + O(y^2), \quad A_1(y) = 1 + \frac{5}{11} \pi y + O(y^2)$$

s_0 is the short-range cutoff, $s_0 \ll S$, necessary in 3D. Note also that in 3D, $y = (2\pi)^{-3} c$ no longer depends on the chain size. The results (7.3) are long-chain asymptotic results, valid for $y \ln(S/s_0) \gg 1$. For short chains, the simple results of perturbation expansion are⁽⁷³⁾ (in dimensional regularization) and at first order

$$R^2 = 3S(1 - 4\pi y) \quad (3D) \quad (7.4)$$

$$\mathcal{Z}(\mathcal{S}_1) = 1 - 4\pi y \quad (3D) \quad (7.4)$$

[At next order, terms $y^2 \ln S/s_0$ appear. Hence, if $y \ln S/s_0 \ll 1$, then (7.4) is valid; and if $y \ln S/s_0 \gg 1$, renormalization calculations^(72,25) lead to (7.3).]

It is worth noting that in 3D, the logarithmic corrections are asymptotically vanishing in R^2 and $\mathcal{Z}(\mathcal{S}_1)$ [see (7.3)], and are thus very difficult to test numerically.⁽⁷⁶⁾ This is quite different from the case of a polymer in a good 4-dimensional solvent, where multiplicative $(\ln S)^{1/4}$ factors appear in R^2 [Eq. (3.43)] and $\mathcal{Z}(\mathcal{S}_1)$ [Eq. (3.47)]. These corrections in 4D have been well checked numerically.⁽⁷⁷⁾ In the case of 3D θ -polymers, only a weak logarithmic singularity predicted^(2,72) for the specific heat has been searched for in numerical simulations.⁽⁷⁶⁾

As we shall see, for *networks* at the Θ point, the situation is much more favorable. As soon as 3-leg vertices appear in a network, they will induce *multiplicative logarithmic correction terms* which could be detected numerically.

Below 3D, new tricritical exponents appear, and it is a present challenge to determine their exact values in *two* dimensions.⁽²⁶⁾ The exponents ν , γ , and ϕ have been calculated for $d = 3 - \varepsilon'$,^(78,25)

$$\begin{aligned} \nu &= \frac{1}{2} \left[1 + \frac{4}{3} \frac{\varepsilon'^2}{(11)^2} + O(\varepsilon'^3) \right] \\ \gamma &= 1 + \frac{5}{(22)^2} \varepsilon'^2 + O(\varepsilon'^3) \\ \phi &= \frac{1}{2} + \frac{3\varepsilon'}{22} + O(\varepsilon'^2) \end{aligned} \tag{7.5}$$

Note that the correction terms in ν and γ start only at order ε^2 . This, of course, is related to the appearance of only additive $1/\ln S$ correction terms in 3D in R^2 and $\mathcal{L}(\mathcal{S}_1)$. As is very well known, and was seen in Section 3.5, for the 4D good solvent, the existence of multiplicative logarithmic factors at an upper critical dimension d_c is equivalent to that of $O(\varepsilon)$ terms in the associated critical exponent for $d = d_c - \varepsilon$.

Now I shall give the renormalization and scaling theory of polymer networks at the Θ point. As mentioned above, I obtain interesting tricritical first-order effects, which are much stronger than those previously known (7.3), and were unsuspected in the literature. Note that a theory similar to this one for polymers has been already proposed⁽²⁸⁾ for describing the mutual three-by-three intersection properties of random walks. There I described the scaling theory of nets of walks which had to cross at some prescribed vertices and, in between, were not allowed to have any three-walk crossings. Here we shall just have to add the self- and two-chain interactions induced by three-body terms.

7.2. Perturbation Expansion

At the theta point, all quantities are calculated from weight (7.1) taken at $b = 0$, and reading for \mathcal{N} chains

$$\begin{aligned} \mathcal{A}_{\mathcal{N}}^{\Theta} \{ \mathbf{r}_a \} &= \frac{1}{2} \sum_{a=1}^{\mathcal{N}} \int_0^S \dot{\mathbf{r}}_a^2(s) ds \\ &+ \frac{c}{3!} \sum_{a=1}^{\mathcal{N}} \sum_{a'=1}^{\mathcal{N}} \sum_{a''=1}^{\mathcal{N}} \int_0^S ds \int_0^S ds' \int_0^S ds'' \delta^d[\mathbf{r}_a(s) - \mathbf{r}_a(s')] \\ &\times \delta^d[\mathbf{r}_{a'}(s') - \mathbf{r}_{a''}(s'')] \end{aligned} \tag{7.6}$$

We shall be interested here in calculating, within this formalism, any polymer partition function \mathcal{Z}_G^θ of a network \mathcal{G} of the type (2.2). For this we shall first need perturbative rules and then a renormalization theory. Both will be very similar to those explained in Sections 2 and 3.

The perturbation expansion of any partition function $\mathcal{Z}^\theta(\mathcal{G})$ is performed in powers of c , and more precisely of the dimensionless y [Eq. (7.2)]. The same dimensional analysis starting from the Brownian value of $\mathcal{Z}^B(\mathcal{G})$ for $c=0$ yields the general form of the tricritical partition function $\mathcal{Z}^\theta(\mathcal{G})$,

$$\mathcal{Z}^\theta(\mathcal{G}) = (2\pi S)^{-d\mathcal{L}/2} Z^\theta(\mathcal{G}, y, d) \tag{7.7}$$

where now Z^θ depends on y (and d) only. As usual,⁽²⁵⁾ we work in $d = 3 - \epsilon'$ dimensions, $\epsilon' > 0$. When $\epsilon' \rightarrow 0$, poles appear in the perturbation expansion of $\mathcal{Z}^\theta(\mathcal{G}, y, d)$ in powers of y , and have to be renormalized.

This perturbation expansion is obtained in a way completely similar to that described in Section 2.4 for two-body interactions. The only difference is that now interactions c are represented by dotted lines joining *three* interaction points, which belong to one or several chains. Let us consider the simplest objects, namely the tricritical star partition functions $\mathcal{Z}^\theta(\mathcal{S}_L)$. Their expansion to first order in c is given in Fig. 19. Apart from the appearance of three-point interaction dotted lines, the rules for calculating diagrams are exactly the same as in Section 2.4.

Since there is no constitutive loop in the star, $\mathcal{L} = 0$, and $\mathcal{Z}^\theta(\mathcal{S}_L)$ [see (7.7)] is dimensionless. It reads, according to Fig. 19,

$$\mathcal{Z}^\theta(\mathcal{S}_L) = 1 + L\mathfrak{J}_1^\theta + L(L-1)\mathfrak{J}_2^\theta + \binom{L}{3}\mathfrak{J}_3^\theta + \dots \tag{7.8}$$

where $\mathfrak{J}_{1,2,3}^\theta$ are the individual contributions of the diagrams of Fig. 19. They read, respectively, according to the rules of Section 2.4 and after momentum integrations

$$\mathfrak{J}_1^\theta = -c(2\pi)^{-d} \int_0^\infty ds ds' \theta(S-s-s')(S-s-s') s^{-d/2} s'^{-d/2}$$



Fig. 19. Expansion to first order in the three-body interaction c of the star partition function $\mathcal{Z}^\theta(\mathcal{S}_L)$ at the θ point.

$$\mathfrak{J}_2^\theta = -c(2\pi)^{-d} \int_0^\infty ds ds' \theta(S-s-s') s^{-d/2} \int_0^S ds'' (s'+s'')^{-d/2}$$

$$\mathfrak{J}_3^\theta = -c(2\pi)^{-d} \int_0^S ds \int_0^S ds' \int_0^S ds'' (ss'+s's''+s''s)^{-d/2}$$

It is not difficult to calculate the analytic continuation of the first two integrals in d dimensions and see that they are regular at $d = 3$ when $\epsilon' \rightarrow 0$. Hence, the only diverging integral at this level is \mathfrak{J}_3^θ , which behaves as

$$\mathfrak{J}_3^\theta = -y(2\pi/\epsilon' + \dots), \quad \epsilon' = 3 - d \rightarrow 0 \tag{7.9}$$

So the Taylor–Laurent expansion of $\mathcal{Z}^\theta(\mathcal{S}_L)$ in powers of y and ϵ is simply, to first order in y ,

$$\mathcal{Z}^\theta(\mathcal{S}_L) = 1 - y \binom{L}{3} \frac{2\pi}{\epsilon'} \tag{7.10}$$

Of course, for $L = 1, 2$, we set, by convention, in (7.8) $\binom{L}{3} = 0$. So we see that for $L = 1$ or $L = 2$, which both correspond to a *single-chain* partition function, there is no divergence to first order. In particular, we have from (7.8)

$$\mathcal{Z}^\theta(\mathcal{S}_1) = 1 + \mathfrak{J}_1^\theta = 1 - 4\pi y \tag{3D}$$

in agreement with (7.4). The renormalized correction terms $1/\ln S$ of (7.3) or, equivalently, the $O(\epsilon^2)$ terms in γ [see (7.5)] will come only at next order $O(y^2)$ (see refs. 25 and 72 for precise calculations). Let us now proceed to the direct renormalization of the star partition function (7.10).

7.3. Tricritical Direct Renormalization

7.3.1. Principles. The principles of this *direct* renormalization of the Edwards model at the tricritical θ point have been given in ref. 25. Their equivalence to the field-theoretic renormalization of the ϕ^6 $O(n)$ model was shown in ref. 79. When renormalizing off the θ point for $z \neq 0$ some subtleties appears.⁽²⁵⁾ But, exactly at $z = 0$, the renormalization is quite simple and very similar to that described in Section 3.1 for the good solvent case. We need essentially the end-to-end distance (at θ)

$$R^2 = \mathcal{X}_0^\theta(y, d) dS \tag{7.11}$$

where the swelling factor has the Taylor–Laurent expansion⁽²⁵⁾

$$\mathcal{X}_0^\theta(y, d) = 1 - 4\pi y + y^2 \frac{\pi^2}{\epsilon'} \left(186 + \frac{2}{3} \right) + \dots \tag{7.12}$$

The single-chain partition function reads similarly

$$\mathcal{Z}^\theta(\mathcal{S}_1) = 1 - 4\pi y + y^2 \frac{\pi^2}{\varepsilon'} 186 + \dots \tag{7.13}$$

The effective renormalized theory will be expressed in terms of the renormalized three-body interaction h , instead of y . It is defined as⁽²⁵⁾

$$h = - \frac{\mathcal{Z}(\mathcal{S}_1 \times \mathcal{S}_1 \times \mathcal{S}_1)}{[\mathcal{Z}(\mathcal{S}_1)]^3} \left(\frac{2\pi R^2}{d} \right)^{-d}$$

where $\mathcal{Z}(\mathcal{S}_1 \times \mathcal{S}_1 \times \mathcal{S}_1)$ is the *connected* partition function of three independent chains. Its Taylor-Laurent expansion in powers of y and ε' can be calculated by diagrams and reads

$$h = y - y^2 \frac{44}{\varepsilon'} \pi + \dots \tag{7.14}$$

For the θ point, h plays the role of the parameter g above. It will reach a fixed point value for $d = 3 - \varepsilon'$, when $S \rightarrow \infty$, and tends logarithmically to zero when $d = 3$. These behaviors are obtained immediately from the Wilson function

$$W(y, \varepsilon') = W[h, \varepsilon'] = S \frac{\partial}{\partial S} h = \varepsilon' y \frac{\partial}{\partial y} h$$

It reads, from (7.14),

$$W[h, \varepsilon'] = \varepsilon' h - 44\pi h^2 + O(h^3) \tag{7.15}$$

and is regular to all orders in h , when $\varepsilon' \rightarrow 0$. The nontrivial fixed point is thus, for $d = 3 - \varepsilon'$,

$$h^*(\varepsilon') = \frac{\varepsilon'}{44\pi} + O(\varepsilon'^2) \tag{7.16}$$

while for $d = 3$, $\varepsilon' = 0$, (7.15) is integrated to

$$h = \frac{1}{44\pi \ln S/s_0} + \dots \tag{7.17}$$

The effective scaling behavior of diverging partition functions like (7.12), (7.13), or (7.10) is obtained by introducing scaling functions like

$$\sigma_0(y, \varepsilon') = \sigma_0[h, \varepsilon'] = S \frac{\partial}{\partial S} \ln \mathcal{X}_0^\theta(y, \varepsilon') = \varepsilon' y \frac{\partial}{\partial y} \ln \mathcal{X}_0^\theta(y, \varepsilon')$$

or

$$2\sigma_1(y, \varepsilon') = 2\sigma_1[h, \varepsilon'] = \varepsilon' y \frac{\partial}{\partial y} \ln \mathcal{Z}^\theta(\mathcal{S}_1)$$

These scaling functions, once expressed in terms of h substituted to y , become *regular* to all orders in h , when $\varepsilon' \rightarrow 0$. For $d = 3 - \varepsilon'$, their fixed point values $\sigma_0[h^*, \varepsilon']$ and $2\sigma_1[h^*, \varepsilon']$ give the tricritical exponents $2\nu - 1$ and $\gamma - 1$.

For $\varepsilon' = 0$, one substitutes (7.17) in the above scaling functions and integrates back with respect to S . After some careful calculations, one finds⁽²⁵⁾ the known results (7.3).

7.3.2. Θ -Star Polymers. Let us now apply the same formalism to the star partition function (7.10). This is very easy: we define a scaling function

$$\begin{aligned} \gamma_L(y, \varepsilon') - 1 &\equiv S \frac{\partial}{\partial S} \ln \mathcal{Z}^\Theta(\mathcal{S}_L) = \varepsilon' y \frac{\partial}{\partial y} \ln Z^\Theta \\ &= -y \binom{L}{3} 2\pi + \dots \end{aligned}$$

Hence, in terms of h ,

$$\gamma_L[h, \varepsilon'] - 1 = -2\pi \binom{L}{3} h + O(h^2) \tag{7.18}$$

Hence, the fixed point value (7.16) gives below 3D the tricritical γ -star exponent

$$\gamma_L^\Theta - 1 = -\frac{1}{6} L(L-1)(L-2) \frac{\varepsilon'}{22} + O(\varepsilon'^2) \tag{7.19}$$

In exactly 3D, we find by substituting (7.17) in (7.18) the differential equation

$$S \frac{\partial}{\partial S} \ln \mathcal{Z}^\Theta(\mathcal{S}_L) = -\binom{L}{3} \frac{1}{22 \ln S}$$

which is trivially integrated into the exact result

$$\mathcal{Z}^\Theta(\mathcal{S}_L) \sim (\ln S)^{-L(L-1)(L-2)/6 \times 22} \tag{7.20}$$

This formula is (asymptotically) exact in 3D. We see that for $L \geq 3$, non-trivial logarithmic factors appear, which would be very interesting to test numerically. We are now in a position to calculate the behavior of any Θ polymer network after having renormalized it.

7.4. Renormalization of Θ -Networks

7.4.1. Star Factorization. Consider a network \mathcal{G} in a bulk Θ -solvent with n_L L -leg vertices, $L \geq 1$. Exactly at the Θ point, its partition function reads as in (7.7). We choose to factorize it over its vertices, and write, exactly as in Eq. (3.18) in the good solvent case,

$$\mathcal{Z}^\Theta(\mathcal{G}) = (2\pi R^2/d)^{-d\mathcal{L}/2} \prod_{L \geq 1} [\mathcal{Z}^\Theta(\mathcal{S}_L) \mathcal{Z}^\Theta(\mathcal{S}_1)^{-L/2}]^{n_L} \mathcal{A}^\Theta(\mathcal{G}, y, d) \tag{7.21}$$

where A^Θ is a *dimensionless* amplitude, a function only of y and d (for $d < 3$), and having a Taylor–Laurent series expansion in powers of y , $\epsilon' = 3 - d$. The crucial statement of tricritical direct renormalization⁽⁷⁹⁾ is then that \mathcal{A}^Θ reaches a finite fixed point limit when $S \rightarrow \infty$, or $y \rightarrow \infty$ in $d = 3 - \epsilon'$:

$$\mathcal{A}^\Theta(\mathcal{G}, y \rightarrow \infty, d) = \mathcal{A}^{\Theta^*}(\mathcal{G}, d) < \infty$$

More precisely, the substitution of the renormalized third virial coefficient h of (7.14) to y transforms $\mathcal{A}^\Theta[\mathcal{G}, h, d]$ into a double Taylor series expansion of h and ϵ' , regular when $\epsilon' \rightarrow 0$, to all orders in h :

$$\begin{array}{ccc} \mathcal{A}^\Theta(\mathcal{G}, y, d) & \equiv & \mathcal{A}^\Theta[\mathcal{G}, h, d] \\ \text{singular} & & \text{regular} \end{array}$$

For $d < 3$, one has of course the fixed point limit

$$\mathcal{A}^\Theta \xrightarrow[y \rightarrow \infty]{} \mathcal{A}^\Theta[\mathcal{G}, h^*]$$

For $d = 3$, $\mathcal{A}^\Theta[h, d = 3]$ is a regular series of $h \sim 1/\ln S$ [Eq. (7.17)]

$$\mathcal{A}^\Theta[\mathcal{G}, h, d = 3] = \mathcal{A}^B(\mathcal{G}, d = 3) + O(h) \tag{7.22}$$

starting as the Brownian value of \mathcal{A} , with calculable corrections.

For $d < 3$, we define a set of independent tricritical exponents σ_L^Θ such that

$$\mathcal{Z}^\Theta(\mathcal{S}_L) [\mathcal{Z}^\Theta(\mathcal{S}_1)]^{-L/2} \xrightarrow[y \rightarrow \infty, S \rightarrow \infty]{} y^{(1/\epsilon)\sigma_L^\Theta} \sim S^{\sigma_L^\Theta} \tag{7.23}$$

and from (7.21) we find as usual the basic hyperscaling relation

$$\begin{aligned} \mathcal{Z}^\Theta(\mathcal{G}) &\sim S^{\gamma_\mathcal{G}^\Theta - 1} \quad (S \rightarrow \infty) \\ \gamma_\mathcal{G}^\Theta - 1 &= -v^\Theta d\mathcal{L} + \sum_{L \geq 1} n_L \sigma_L^\Theta \end{aligned} \tag{7.24}$$

which generalizes Eq. (3.34) at the Θ point and in any dimension $d < 3$.

The value of σ_L^Θ is immediately found to first order in $\varepsilon' = 3 - d$ from (7.19),

$$\begin{aligned} \sigma_L^\Theta &= \gamma_L^\Theta - 1 - \frac{L}{2}(\gamma^\Theta - 1) \\ &= -\frac{1}{6}L(L-1)(L-2)\frac{\varepsilon'}{22} + O(\varepsilon'^2) \end{aligned} \tag{7.25}$$

In three dimensions, the problem is exactly solvable. Indeed, we plug results (7.3), (7.20), and (7.22) into (7.21), and find

$$\mathcal{L}^\Theta(\mathcal{G}, d=3) \underset{S \rightarrow \infty}{\sim} S^{-3\mathcal{L}^\Theta/2} (\ln S)^{-\sum_{L \geq 1} n_L L(L-1)(L-2)/132} \tag{7.26}$$

which is asymptotically exact. This completes our study of networks in a Θ -solvent.

7.4.2. Two Dimensions. In another work,⁽²⁶⁾ exact values have been proposed for the exponents of the Θ point in *two dimensions*. The critical exponents were those of the $O(n=1)$ model in its critical low-temperature phase, or, equivalently,^(26,20,80) of the $Q=1$ Potts model at its critical point. The model proposed was a model of self-avoiding walks on the hexagonal lattice, in the presence of random forbidden hexagons, with annealed randomness. At the percolation threshold of the hexagons (which is the site percolation of their centers on the triangular dual lattice, at $p_c = 1/2$), the sudden restriction of the space available to the SAW leads to its collapse. Note that the Θ point is frequently defined as the point where nearest-neighbor (nn) attractive two-body interactions just counterbalance the excluded-volume effects. In the percolation forbidden hexagon model, the collapse is driven by the annealed impurities. There is some discussion at present⁽⁸¹⁻⁸³⁾ about the universality class of this transition and about its equivalence to the “standard” Θ point. It is interesting, to remark that it is possible to show this equivalence in dimension $d > 2$,⁽⁸⁴⁾ but new instabilities appear in 2D and several “ Θ points” could be possible there.

In any case, the above annealed disorder model has been solved exactly in 2D by Coulomb-gas techniques, and the critical exponents are at the collapse transition in the bulk⁽²⁶⁾:

$$x_L^\Theta = (L^2 - 1)/12 \tag{7.27}$$

and read in terms of the Kac table (4.17) $x_L^\Theta = 2h_{L,L}$ for $m=2$. The surface exponents, valid for the collapse transition near a Dirichlet surface (ordinary transition), are also known,⁽²⁶⁾

$$x_L^{S\Theta} = L(L-1)/6 = h_{1,L+1} \quad (m=2) \tag{7.28}$$

The former scaling theory above applied integrally to this special two-dimensional Θ polymer system. The usual exponents ν and γ and crossover exponent ϕ are⁽²⁶⁾

$$\begin{aligned} \nu^{-1} &= 2 - x_2, & \gamma &= (2 - 2x_1)\nu \\ \phi &= \nu/\nu', & \nu'^{-1} &= 2 - x_4 \end{aligned} \quad (7.29)$$

giving

$$\nu = 4/7, \quad \gamma = 8/7, \quad \nu' = 4/3, \quad \phi = 3/7 \quad (7.30)$$

These values have been studied numerically for the obstacle model on strips⁽²⁶⁾ and in the usual model of a Θ -solvent with attractive nn interactions by Monte Carlo methods.^(85,81)

We can now consider a two-dimensional Θ -network (Θ in the sense of the annealed obstacle model). Then, the irreducible σ_L^Θ exponents are deduced from the scaling x_L^Θ by the usual scaling rules (3.53), which apply universally,

$$\begin{aligned} \sigma_L^\Theta &= -\nu x_L^\Theta + (\nu - 1/2)L \\ &= -(2L + 1)(L - 2)/42 \end{aligned} \quad (7.31)$$

Similarly, for surface exponents [Eq. (6.66) in $d=2$]

$$\begin{aligned} \sigma_L^{S\Theta} &= -\nu x_L^{S\Theta} + (\nu - 1/2)L \\ &= (31L - 4L^2)/42 \end{aligned} \quad (7.32)$$

The $\gamma_{\mathcal{G}}^\Theta$ exponent of a network under Θ conditions is then found from the equivalent equations (7.24) or (4.15), and near a surface from Eq. (6.6) generalized to the Θ point. Using the above exact values gives explicitly⁽²⁶⁾

$$\gamma_{\mathcal{G}}^\Theta - 1 = -\frac{8}{7} + \frac{1}{42} \sum_{L \geq 1} n_L (2 - L)(2L + 25) \quad (7.33)$$

and

$$\gamma_{\mathcal{G}}^{S\Theta} - 1 = -\frac{4}{7} + \frac{1}{42} \sum_{L \geq 1} [n_L (2 - L)(2L + 25) + n_L^S (24 - 4L^2 - 17L)] \quad (7.34)$$

Finally, the proximal surface exponents ζ_L^Θ of (6.103) read at the Θ point

$$\zeta_L^\Theta = x_L^{S\Theta} - x_L^\Theta = (L - 1)^2/12 \quad (7.35)$$

8. EXACT EXPONENTS IN TWO DIMENSIONS. COMPARISON TO NUMERICAL ESTIMATES

I have given a general hyperscaling theory, valid in any dimension, which predicts the critical exponents for any geometry of polymer networks in terms of a basic set of exponents (σ_L or x_L) associated with the L -leg vertices floating in the bulk solvent or in terms of exponents σ_L^S , x_L^S in the case of the surface (ordinary) transition. Such a theory was originally imagined by observing that the renormalization of any polymer network could be performed in terms of the L -vertex operators appearing in an L -watermelon, as described in Section 3. Then the general hyperscaling theory was exploited extensively in *two dimensions* in refs. 11, 13, 18, 22, and 26, and to $O(\varepsilon^2)$ in $d=4-\varepsilon$ for the intersections of networks of random walks.⁽²⁸⁾ It is worth noting that there is up to now no rigorous proof of the validity of this generalized hyperscaling theory to all orders in renormalization theory.

I stress that here and in refs. 1 and 28 use was made of *direct* renormalization theory, where polymer partition functions are calculated directly. Now, the polymer theory, including vertices of higher order, can be identically transformed into a field theory, where L -leg vertices correspond essentially to insertions of composite operators φ^L . Such a transformation was performed here for the basic watermelon network (Sections 3.6 and 4) and it was actually at the root of all our studies in two dimensions, where the $O(n)$ model formulation was used intensively (see, in particular, refs. 13, 17, and 20). Then, knowing this equivalence polymer- $O(n)$ model, the *direct* renormalization method for simple linear polymer chains originally devised by des Cloizeaux in the good solvent case⁽⁵⁾ can be shown to be completely equivalent to the renormalization of the usual $(\varphi^2)^2$ theory^(34,35,44) (for $n=0$) and can be extended to Θ -solvents.⁽⁷⁹⁾ For highly branched polymers, one would have to consider (in polymer as well as in field theory) the full renormalization of composite operators. In field theory it is known in principle⁽¹⁵⁾ but quite complicated because of the possible mixing of many operators.

My statement is that it can be understood from a basic series of anomalous dimensions x_L (or σ_L), corresponding to irreducible scaling operators. This is easily seen in polymer theory, but works also in any $O(n)$ theory, as seen in two dimensions.^(20,22) It is also a quite natural idea in the framework of two-dimensional conformal invariance.^(13,17,20)

Having established these scaling relations, I feel it useful to convince the reader of their validity by comparing the predictions to numerical estimates. The best test is of course in *two dimensions*, where all exact values are known, and where good series and Monte Carlo numerical results are available.

Furthermore, the exact results in 2D should not be thought to be derived only from conformal invariance. More precisely, only the values of the watermelon critical exponents x_L or x_L^S are found in the conformal table. All other geometrical exponents are derived from the multiplicative renormalization (3.18) over vertices, equivalent to the scaling theory of Section 4.

Let us briefly recall the exact 2D exponents.

First, one needs to know the scaling laws giving the $\gamma_{\mathcal{G}}$ exponents. They are, in the bulk,⁽¹⁾ (4.14)

$$\gamma_{\mathcal{G}} = \nu \left[d(\mathcal{V} - 1) - \sum_{L \geq 1} n_L x_L \right] - (\mathcal{N} - 1)$$

and near a Dirichlet surface⁽²²⁾ (6.6)

$$\gamma_{\mathcal{G}}^S = \nu \left[d\mathcal{V} + (d-1)(\mathcal{V}_S - 1) - \sum_{L \geq 1} (n_L x_L + n_L^S x_L^S) \right] - (\mathcal{N} - 1)$$

In $d=2$, one has the Nienhuis' value $\nu = 3/4$.⁽¹⁶⁾ Recall that the x_L , x_L^S exponents (2.40) and (6.4) are η -like exponents of the critical correlation functions (2.36) and (6.4) of the peculiar watermelon network. In 2D they belong to the conformal Kac table (4.17).⁽⁴⁵⁾ In the bulk good solvent case, their exact values, (4.18),

$$x_L = (9L^2 - 4)/48 \quad (8.1)$$

were originally conjectured⁽¹²⁾ from numerical simulations followed by identifications to the Kac table of central charge $c=0$.⁽²⁷⁾ So these values can be considered from the start as well checked numerically. Furthermore, they can be derived in the Coulomb-gas formalism, and are exact beyond all doubt. In a similar way the surface exponents in 2D, $x_L^S = L(3L+2)/8$, were originally found from numerical simulations on strips,⁽²²⁾ and, again, identification in the $c=0$ Kac table, with $x_L^S = h_{L+1,1}$.

The associated exponents σ_L or A_L [(3.53), (3.54)] are thus, in 2D,

$$\sigma_L = \frac{1}{64} (2-L)(9L+2), \quad A_L = \frac{1}{64} (2-L)(9L+50) \quad (8.2)$$

The configuration exponent of a general network \mathcal{G} , (3.34bis), is then

$$\gamma_{\mathcal{G}} = -\frac{1}{2} + \frac{1}{64} \sum_{L \geq 1} n_L (2-L)(9L+50) \quad (8.3)$$

In particular, for stars in the bulk, one finds⁽¹⁾

$$\gamma_{\mathcal{S}_L} = 1 + [4 + 9L(3 - L)]/64 \tag{8.4}$$

The first values are especially interesting, since series enumerations were performed independently for them⁽⁶³⁾: they are

$$\begin{aligned} \gamma_{\mathcal{S}_1} = \gamma_{\mathcal{S}_2} \equiv \gamma &= \frac{43}{32}, & \gamma_{\mathcal{S}_3} &= \frac{17}{16}, & \gamma_{\mathcal{S}_4} &= \frac{1}{2}, \\ \gamma_{\mathcal{S}_5} &= -\frac{11}{32}, & \gamma_{\mathcal{S}_6} &= -\frac{47}{32} \dots \end{aligned} \tag{8.5}$$

Also, the *H*-network exponent will be interesting,⁽⁸⁶⁾ corresponding to $n_1 = 4, n_3 = 2$. Its value is

$$\gamma_H = 25/32 \tag{8.6}$$

As explained above, all contact exponents can be calculated. In particular, the well-known exponents θ_1, θ_2 are⁽¹¹⁾

$$\theta_1 = 5/6, \quad \theta_2 = 19/12 \tag{8.7}$$

The “limiting ring closure probability” exponent (5.27) governing the probability $P \sim S^{-\gamma_1}$ of forming a tadpole^(52,53) is then⁽¹¹⁾

$$\gamma_1 = (2 + \theta_1)\nu = 17/8 \tag{8.8}$$

It will also be very interesting to check the general surface exponents. The basic surface scaling dimensions x_L^S , (6.4), are in $2D^{(22)}$ in terms of the Kac formula (4.17) for $m = 2$

$$x_L^S = h_{L+1,1} = (3L + 2)L/8 \tag{8.9}$$

or in the “chain size space” [Eq. (6.66)]

$$\sigma_L^S = \frac{L}{32} (-9L + 26) \tag{8.9bis}$$

The total vertex contributions A_L^S , (6.64), are then

$$A_L^S = -\frac{1}{32} (9L^2 + 22L - 24) \tag{8.9ter}$$

As a result, the general surface exponent $\gamma_{\mathcal{G}}^S$ of any grafted network \mathcal{G} reads⁽²²⁾ [Eq. (6.63)]

$$\gamma_{\mathcal{G}}^S = \frac{1}{4} + \frac{1}{64} \sum_{L \geq 1} n_L (2-L)(9L+50) - \frac{1}{32} \sum_{L \geq 1} n_L^S (9L^2 + 22L - 24) \quad (8.10)$$

Particular cases are the usual magnetic surface exponents γ_1 and γ_{11} , which correspond to $n_1^S = 1$, $n_1 = 1$ (γ_1), and $n_1^S = 2$ (γ_{11}). Their values are

$$\gamma_1 = 61/64, \quad \gamma_{11} = -3/16 \quad (8.11)$$

in agreement with Cardy,⁽⁶⁰⁾ Guttman and Torrie,⁽⁸⁷⁾ and Barber's law.

Polymers grafted in a wedge also present very interesting effects.^(65,87,22) One expects indeed new scaling dimensions, depending on the wedge angle α .⁽⁸⁸⁾ In two dimensions, the conformal mapping $w(z) = z^{\pi/\alpha}$ transforms the wedge α into a surface line and by conformal invariance one easily shows^(27,22) that the anomalous scaling dimensions of surface operators ϕ_L^S , (6.4), become in the wedge

$$x_L^W(\alpha) = \frac{\pi}{\alpha} x_L^S \quad (8.12)$$

Originally, a single chain grafted inside a wedge by one of its extremities was considered. Cardy and Redner⁽⁶⁵⁾ and Guttman and Torrie⁽⁸⁷⁾ independently found the wedge-dependent exponent

$$\gamma_{\bar{1}}(\alpha) = \frac{91}{64} - \frac{15}{32} \frac{\pi}{\alpha} \quad (8.13)$$

and the formula for a single chain grafted at one end into the wedge and at the other on the surface was also guessed numerically,⁽⁸⁷⁾

$$\gamma_{11}(\alpha) = \frac{9}{32} - \frac{15}{32} \frac{\pi}{\alpha} \quad (8.14)$$

Note that, as expected for $\alpha = \pi$, one recovers $\gamma_{\bar{1}}(\pi) \equiv \gamma_1$ and $\gamma_{1,1}(\pi) = \gamma_{11}$. In ref. 22, this was completely generalized to any network \mathcal{G} , one \hat{L} -vertex of which is grafted inside the wedge of angle α , while other $\{n_L\}$ bulk L -vertices float in the solvent, and $\{n_L^S\}$ surface L -vertices are grafted along the surfaces. The wedge corresponding configuration exponent is then simply related to the surface exponent⁽²²⁾ by

$$\gamma_{\mathcal{G}}^W(\alpha) = \gamma_{\mathcal{G}}^S(\pi) - v(\pi/\alpha - 1) x_{\hat{L}}^S \quad (8.15)$$

where one recognizes the wedge contribution (8.12). The explicit value is then⁽²²⁾

$$\begin{aligned} \gamma_{\mathcal{G}}^W(\alpha) - 1 = & -\frac{1}{2} \hat{L} - \frac{3}{32} \frac{\pi}{\alpha} \hat{L}(3\hat{L} + 2) \\ & + \sum_{L \geq 1} \left[\frac{1}{64} n_L(2-L)(9L+50) - \frac{1}{32} n_L^S(9L^2 + 22L - 24) \right] \end{aligned} \tag{8.16}$$

For $\hat{L} = \hat{1}$ and $n_1 = 1$ or $n_1^S = 1$ one recovers (8.13) or (8.14).

Some simple geometries have been analyzed by series enumeration by Colby *et al.*⁽⁶⁶⁾ for checking the exact 2D surface and wedge predictions (8.10), (8.16) (Fig. 20).

Let us consider an L -star grafted by its core on the surface or in a wedge ($n_1 = L$, $n_L^S = 1$ or $n_1 = L$, $n_L^S = 0$, $\hat{L} = L$). Then

$$\gamma_L^S = 1 + L(15 - 18L)/64 \tag{8.17}$$

$$\gamma_1^S = 61/64, \quad \gamma_2^S = 11/32, \quad \gamma_3^S = -53/64, \dots$$

$$\gamma_L^W(\alpha) - 1 = 27L/64 - 3\pi L(3L + 2)/32\alpha \tag{8.18}$$

Of course, for $\alpha = \pi$, $\gamma_L^W(\pi) \equiv \gamma_L^S$.

Also of numerical interest⁽⁶⁶⁾ will be the L -star tied to the surface by M arms⁽²²⁾

$$\gamma_{L,M} = 29/16 + 9L(3 - L)/64 - 73M/64 \tag{8.19}$$

For $L = 3$, $M = 1$, $\gamma_{3,1} = 43/64$.

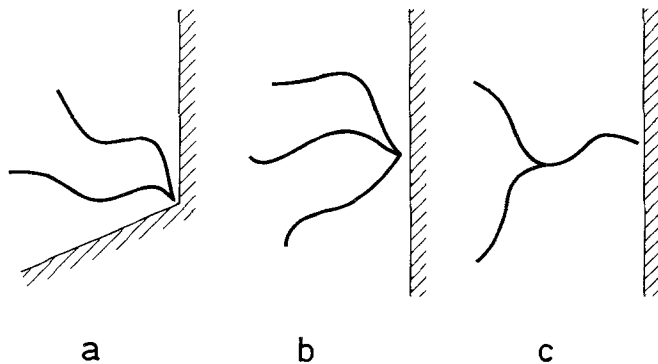


Fig. 20. Stars in wedge or surface geometries, with exponents (a) $\gamma_2^W(\alpha)$, (b) $\gamma_3^S(\pi)$, and (c) $\gamma_{3,1}^S$.

Table I. Comparison of Exact 2D Predictions (8.5)–(8.8), (8.11), (8.18), and (8.19) with Values Found Numerically

L-Stars	Exact	Numerical
γ_3	$17/16 = 1.0625^{(1)}$	$1.07 \pm 0.02^{(63)}$
γ_4	$1/2^{(1)}$	$0.52 \pm 0.04^{(63)}$
γ_5	$-11/32 = -0.34375^{(1)}$	$-0.29 \pm 0.04^{(63)}$
γ_6	$-47/32 = -1.46875^{(1)}$	$-1.33 \pm 0.05^{(63)}$
<i>H-Comb</i>		
γ_H	$25/32 = 0.78125^{(1)}$	$0.79 \pm 0.02^{(86)}$
$Y = (2 + \theta_1)\nu$	$17/8 = 2.125^{(11)}$	$2.13 \pm 0.01^{(52)}$
		$2.10 \pm 0.10^{(53)}$
		$2.15 \pm 0.30 - 0.15^{(53)}$
Contact exponents		
θ_1	$5/6 = 0.8333...^{(11)}$	$0.84 \pm 0.01^{(52)}$
		$0.84 \pm 0.13^{(53)}$
θ_2	$19/12 = 1.5833...^{(11)}$	$1.93 \pm 0.27^{(89)}$
Surface and wedge exponents		
γ_1	$61/64 = 0.9531...^{(60,87)}$	$0.945 \pm 0.005^{(64)}$
		$0.956 + 0.014^{(90)} - 0.006$
$\gamma_2^W(\pi/2)$	$-37/32 = -1.15625^{(22)}$	$-1.15 \pm 0.05^{(66)}$
$\gamma_2^W(2\pi/3)$	$-13/32 = -0.40625^{(22)}$	$-0.4 \pm 0.05^{(66)}$
$\gamma_2^W(\pi)$	$11/32 = 0.34375^{(22)}$	$0.35 \pm 0.05^{(66)}$
$\gamma_3^S(\pi)$	$-53/64 = -0.828^{(22)}$	$-0.82 \pm 0.05^{(66)}$
$\gamma_{3,1}^S(\pi)$	$43/64 = 0.671^{(22)}$	$0.68 \pm 0.05^{(66)}$

Table I collects the exact 2D predictions (8.5)–(8.8), (8.11), (8.18), and (8.19) with the values found numerically. The agreement is excellent. Slight discrepancies, such as in γ_5 , γ_6 , or θ_2 , should not be taken too seriously, since according to the authors themselves,^(63,89) the lengths of the chains were too short. This agreement fully confirms the validity of the above polymer network scaling theory, and also in two dimensions of the conformal invariance approach.

9. CONCLUSION

In conclusion, the scaling theory of uniform (and also of polydisperse) polymer networks of arbitrary topology is now known. The basic idea is that a polymer network can be decomposed onto its vertices, each of these generating its anomalous scaling dimension [Eqs. (1.11), (4.14), (6.6), (6.61), (6.63), (7.24)]. This applies to good and Θ -solvents, and in bulk and semi-infinite geometries. From this scaling theory, infinities of iden-

tities between critical exponents can be obtained by recomposing the same vertices to form other topological networks. Only denumerable sets of basic irreducible exponents exist for each physical situation, which are those of the L -leg polymer vertices, $L \geq 1$: σ_L for the bulk vertices, σ_L^S for surface ones (at the ordinary transition), σ_L^Θ at the Θ point. These exponents give the power law dependence of partition functions as functions of the length of the polymer. In field theory, they are in direct correspondence to η -like exponents x_L , x_L^S , x_L^Θ associated with the critical decay of the correlation function of L polymer lines of fluctuating lengths. All other geometrical exponents can be expressed in terms of the σ 's (or the x 's). The values of the σ_L , σ_L^S , σ_L^Θ are known, respectively, to orders $O(\varepsilon^2)$, $O(\varepsilon)$ in $d=4-\varepsilon$ and $O(\varepsilon')$ in $d=3-\varepsilon'$. The corresponding logarithmic corrections in $d=4$ or $d=3$ for the tricritical case have been given. In two dimensions, all geometrical exponents are known exactly, since conformal invariance and Coulomb-gas techniques give access to the exact x_L , x_L^S , and then by the scaling theory to any γ_φ , γ_φ^S . The comparison of these predictions with existing numerical data in 2D is fairly good. It would be interesting to test also the exact tricritical logarithmic laws predicted by, e.g., Eq. (7.20) for stars in a Θ -solvent. Further progress could be made by extending without difficulty the present theory to networks adsorbed at a surface (i.e., at the special transition point).

A technical study would also be necessary to derive the multiplicative renormalization of polymer vertices from the rigorous renormalization of composite operators of any order in field theory. Note also that my feeling is that the present renormalization over vertices is not specific to polymers, but works in the $O(n)$ model. Multiply connected magnetic correlation functions could be studied there. In 2D, this already appeared in the $O(n)$ model, in some exact applications to geometrical critical phenomena.⁽²⁶⁾

Note added. After this work was completed, I learned of a preprint by K. Ohno and K. Binder where the present scaling theory, originally given in refs. 1 and 22 and extended here, is rederived using a phenomenological scaling in the framework of the $O(n)$ model ($n=0$). The values of the ε expansions of the exponents given there agree with ours.

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